CIMPA SCHOOL, 2007

Jump Processes and Applications to Finance

Monique Jeanblanc

Jump Processes

I. Poisson Processes

II. Lévy Processes

III. Jump-Diffusion Processes

IV. Point Processes

I. Poisson Processes

1. Counting Processes and Stochastic Integral
2. Standard Poisson Process
3. Inhomogeneous Poisson Processes
4. Stochastic Intensity Poisson Processes
5. Compound Poisson Processes

1 Counting Processes and Stochastic Integral

Let T_n be an increasing sequence of random times.

$$N_t = \begin{cases} n & \text{if } t \in [T_n, T_{n+1}[\\ +\infty & \text{otherwise} \end{cases}$$

or, equivalently

$$N_t = \sum_{n \geq 1} \mathbb{1}_{\{T_n \leq t\}} = \sum_{n \geq 1} n \mathbb{1}_{\{T_n \leq t < T_{n+1}\}} .$$

We denote by N_{t-} the left-limit of N_s when $s \to t, s < t$ and by $\Delta N_s = N_s - N_{s-}$ the jump process of N.

The stochastic integral

$$\int_0^t C_s dN_s$$

is defined as

$$(C \star N)_t = \int_0^t C_s dN_s = \int_{]0,t]} C_s dN_s = \sum_{n=1}^{\infty} C_{T_n} \mathbb{1}_{\{T_n \le t\}} .$$

2 Standard Poisson Process

2.1 Definition

The standard Poisson process is a counting process such that

- for every s, t, $N_{t+s} - N_t$ is independent of $\mathcal{F}_t^N$,

- for every s, t, the r.v. $N_{t+s} - N_t$ has the same law as N_s.

Then,

$$P(N_t = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$$

2.2 First properties

- $E(N_t) = \lambda t, \quad \text{Var}\,(N_t) = \lambda t$
- for every $x > 0,\ t > 0, u, \alpha \in I\!R$

$$E(x^{N_t}) = e^{\lambda t(x-1)}\,;\ E(e^{iuN_t}) = e^{\lambda t(e^{iu}-1)}\,;\ E(e^{\alpha N_t}) = e^{\lambda t(e^{\alpha}-1)}\,.$$

2.3 Martingale properties

For each $\alpha \in \mathbb{R}$, for each bounded Borel function h, the following processes are **F**-martingales:

$$(i) \quad M_t = N_t - \lambda t,$$

$$(ii) \quad M_t^2 - \lambda t = (N_t - \lambda t)^2 - \lambda t,$$

$$(iii) \quad \exp(\alpha N_t - \lambda t(e^\alpha - 1)),$$

$$(iv) \quad \exp\Big[\int_0^t h(s)dN_s - \lambda \int_0^t (e^{h(s)} - 1)ds\Big]\,.$$

For any $\beta > -1$, any bounded Borel function h, and any bounded Borel function φ valued in $]-1, \infty[$, the processes

$$\begin{aligned}
&\exp[\ln(1+\beta)N_t - \lambda\beta t] = (1+\beta)^{N_t} e^{-\lambda\beta t}, \\
&\exp\Big[\int_0^t h(s)dM_s + \lambda\int_0^t (1+h(s)-e^{h(s)})ds\Big], \\
&\exp\Big[\int_0^t \ln(1+\varphi(s))dN_s - \lambda\int_0^t \varphi(s)ds\Big], \\
&\exp\Big[\int_0^t \ln(1+\varphi(s))dM_s + \lambda\int_0^t (\ln(1+\varphi(s)) - \varphi(s)ds)\Big],
\end{aligned}$$

are martingales.

Let H be an $\mathbf{F}$-predictable bounded process, then the following processes are martingales

$$(H{\star}M)_t := \int_0^t H_s dM_s = \int_0^t H_s dN_s - \lambda \int_0^t H_s ds$$
$$((H{\star}M)_t)^2 - \lambda \int_0^t H_s^2 ds$$
$$\exp\left(\int_0^t H_s dN_s + \lambda \int_0^t (1 - e^{H_s}) ds\right)$$

These properties does not extend to adapted processes H. For example, from $\int_0^t (N_s - N_{s-}) dM_s = N_t$, it follows that $\int_0^t N_s dM_s$ is not a martingale.

Remark 2.1 Note that (i) and (iii) imply that the process $(M_t^2 - N_t; t \geq 0)$ is a martingale.

The process λt is the predictable quadratic variation $\langle M \rangle$, whereas the process $(N_t, t \geq 0)$ is the optional quadratic variation $[M]$.

For any $\mu \in [0,1]$, the processes $M_t^2 - (\mu N_t + (1-\mu)\lambda t)$ are also martingales.

2.4 Infinitesimal Generator

The Poisson process is a Lévy process, hence a Markov process, its infinitesimal generator $\mathcal{L}$ is defined as

$$\mathcal{L}(f)(x) = \lambda[f(x+1) - f(x)]\,.$$

Therefore, for any bounded Borel function f, the process

$$C_t^f = f(N_t) - f(0) - \int_0^t \mathcal{L}(f)(N_s)ds$$

is a martingale.

Furthermore,

$$C_t^f = \int_0^t [f(N_{s-}+1) - f(N_{s-})]dM_s\,.$$

Exercise:

Extend the previous formula to functions $f(t,x)$ defined on $\mathbb{R}^+ \times \mathbb{R}$ and C^1 with respect to t, and prove that if

$$L_t = \exp(\log(1+\phi)N_t - \lambda\phi t)$$

then

$$dL_t = L_{t-}\phi dM_t\,.$$

2.4.1 Watanabe's characterization

Let N be a counting process and assume that there exists $\lambda > 0$ such that $M_t = N_t - \lambda t$ is a martingale. Then N is a Poisson process with intensity λ.

2.5 Change of Probability

If N is a Poisson process, then, for $\beta > -1$,

$$L_t = (1+\beta)^{N_t} e^{-\lambda \beta t}$$

is a strictly positive martingale with expectation equal to 1.

Let Q be the probability defined as $\frac{dQ}{dP}|_{\mathcal{F}_t} = L_t$.

The process N is a Q-Poisson process with intensity equal to $(1+\beta)\lambda$.

2.6 Hitting Times

Let $T_x = \inf\{t, N_t \geq x\}$. Then, for $n \leq x < n+1$, T_x is equal to the time of the n^{th}-jump of N, hence has a Gamma (n) law.

3 Inhomogeneous Poisson Processes

3.1 Definition

Let λ be an $\mathbb{R}^+$-valued function satisfying $\int_0^t \lambda(u)du < \infty, \forall t$.

An inhomogeneous Poisson process N with intensity λ is a counting process with independent increments which satisfies for $t > s$

$$P(N_t - N_s = n) = e^{-\Lambda(s,t)} \frac{(\Lambda(s,t))^n}{n!}$$

where $\Lambda(s,t) = \Lambda(t) - \Lambda(s) = \int_s^t \lambda(u)du$, and $\Lambda(t) = \int_0^t \lambda(u)du$.

For any real numbers u and α, for any t

$$\begin{aligned} E(e^{iuN_t}) &= \exp((e^{iu}-1)\Lambda(t)) \\ E(e^{\alpha N_t}) &= \exp((e^{\alpha}-1)\Lambda(t)) . \end{aligned}$$

3.2 Martingale Properties

Let N be an inhomogeneous Poisson process with deterministic intensity λ. The process

$$(M_t = N_t - \int_0^t \lambda(s)ds,\ t \geq 0)$$

is an $\mathbf{F}^N$-martingale, and the increasing function $\Lambda(t) = \int_0^t \lambda(s)ds$ is called the compensator of N.

Let ϕ be an $\mathbf{F}^N$- predictable process such that $E(\int_0^t |\phi_s|\lambda(s)ds) < \infty$ for every t. Then, the process $(\int_0^t \phi_s dM_s, t \geq 0)$ is an $\mathbf{F}^N$-martingale.

In particular,

$$E\left(\int_0^t \phi_s\, dN_s\right) = E\left(\int_0^t \phi_s \lambda(s) ds\right) .$$

Let H be an $\mathbf{F}^N$- predictable process. The following processes are martingales

$$(H{\star}M)_t = \int_0^t H_s dM_s = \int_0^t H_s dN_s - \int_0^t \lambda(s) H_s ds$$

$$((H{\star}M)_t)^2 - \int_0^t \lambda(s) H_s^2 ds$$

$$\exp\left(\int_0^t H_s dN_s - \int_0^t \lambda(s)(e^{H_s} - 1) ds\right) .$$

3.3 Watanabe's Characterization

Proposition 3.1 (Watanabe characterization.) *Let N be a counting process and Λ an increasing, continuous function with zero value at time zero. Let us assume that $M_t = N_t - \Lambda(t)$ is a martingale. Then N is an inhomogeneous Poisson process with compensator Λ.*

3.4 Stochastic calculus

3.4.1 Integration by parts formula

Let $X_t = x + \int_0^t x_s dN_s$ and $Y_t = y + \int_0^t y_s dN_s$, where x and y are predictable processes.

$$\begin{aligned} X_t Y_t &= xy + \sum_{s \le t} \Delta(XY)_s = xy + \sum_{s \le t} Y_{s-} \Delta X_s + \sum_{s \le t} X_{s-} \Delta Y_s + \sum_{s \le t} \Delta X_s \, \Delta Y_s \\ &= xy + \int_0^t X_{s-} dY_s + \int_0^t Y_{s-} dX_s + [X, Y]_t \end{aligned}$$

where

$$[X, Y]_t = \sum_{s \le t} \Delta X_s \, \Delta Y_s = \sum_{s \le t} x_s \, y_s \Delta N_s = \int_0^t x_s \, y_s dN_s \, .$$

If $dX_t = \mu_t dt + x_t dN_t$ and $dY_t = \nu_t dt + y_t dN_t$, one gets

$$X_t Y_t = xy + \int_0^t Y_{s-} dX_s + \int_0^t Y_{s-} dX_s + [X, Y]_t$$

where

$$[X, Y]_t = \int_0^t x_s \, y_s dN_s \, .$$

If $dX_t = x_t dM_t$ and $dY_t = y_t dM_t$, the process $X_t Y_t - [X, Y]_t$ is a martingale.

3.4.2 Itô's Formula

Let N be a Poisson process and f a bounded Borel function. The decomposition

$$f(N_t) = f(N_0) + \sum_{0<s\le t} [f(N_s) - f(N_{s-})]$$

is trivial and corresponds to Itô's formula for a Poisson process.

$$\begin{aligned}\sum_{0<s\le t} [f(N_s) - f(N_{s-})] &= \sum_{0<s\le t} [f(N_{s-}+1) - f(N_{s-})]\Delta N_s \\ &= \int_0^t [f(N_{s-}+1) - f(N_{s-})] dN_s \,.\end{aligned}$$

Let

$$X_t = x + \int_0^t x_s dN_s = x + \sum_{T_n \le t} x_{T_n} ,$$

with x a predictable process.

The trivial equality

$$F(X_t) = F(X_0) + \sum_{s \le t} (F(X_s) - F(X_{s-})) ,$$

holds for any bounded function F.

Let

$$dX_t = \mu_t dt + x_t dM_t = (\mu_t - x_t\lambda(t))dt + x_t dN_t$$

and $F \in C^{1,1}(\mathbb{R}^+ \times \mathbb{R})$. Then

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_{s-})(\mu_s - x_s\lambda(s)) ds \\ &\quad + \sum_{s \le t} F(s, X_s) - F(s, X_{s-}) \qquad (3.1) \\ &= F(0, X_0) + \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_{s-}) dX_s \\ &\quad + \sum_{s \le t} \left[F(s, X_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-}) x_s \Delta N_s \right] . \end{aligned}$$

The formula (3.1) can be written as

$$
\begin{aligned}
F(t, X_t) \quad - \quad F(0, X_0) &= \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_s)(\mu_s - x_s \lambda(s)) ds \\
&\quad + \int_0^t [F(s, X_s) - F(s, X_{s-})] dN_s \\
&= \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_{s-}) dX_s \\
&\quad + \int_0^t [F(s, X_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-}) x_s] dN_s \\
&= \int_0^t \partial_t F(s, X_s) ds + \int_0^t \partial_x F(s, X_{s-}) dX_s \\
&\quad + \int_0^t [F(s, X_{s-} + x_s) - F(s, X_{s-}) - \partial_x F(s, X_{s-}) x_s] dN_s \, .
\end{aligned}
$$

3.5 Predictable Representation Property

Proposition 3.2 *Let $\mathbf{F}^N$ be the completion of the canonical filtration of the Poisson process N and $H \in L^2(\mathcal{F}_\infty^N)$, a square integrable random variable. Then, there exists a predictable process h such that*

$$H = E(H) + \int_0^\infty h_s dM_s$$

and $E(\int_0^\infty h_s^2 ds) < \infty$.

3.6 Independent Poisson Processes

Definition 3.3 *A process* $(N^1, \cdots, N^d)$ *is a d-dimensional* $\mathbf{F}$*-Poisson process if each* N^j *is a right-continuous adapted process such that* $N_0^j = 0$ *and if there exists constants* λ_j *such that for every* $t \geq s \geq 0$

$$P\left[\cap_{j=1}^d (N_t^j - N_s^j = n_j)|\mathcal{F}_s\right] = \prod_{j=1}^d e^{-\lambda_j(t-s)} \frac{(\lambda_j(t-s))^{n_j}}{n_j!} .$$

Proposition 3.4 *An* $\mathbf{F}$*-adapted process* N *is a d-dimensional* $\mathbf{F}$*-Poisson process if and only if*

(i) each N^j *is an* $\mathbf{F}$*-Poisson process*

(ii) no two N^j*'s jump simultaneously.*

4 Stochastic intensity processes

4.1 Definition

Definition 4.1 *Let N be a counting process, $\mathbf{F}$-adapted and $(\lambda_t, t \geq 0)$ a non-negative $\mathbf{F}$- progressively measurable process such that for every t, $\Lambda_t = \int_0^t \lambda_s ds < \infty$ P a.s..*
The process N is an inhomogeneous Poisson process with stochastic intensity λ if for every non-negative $\mathbf{F}$-predictable process $(\phi_t, t \geq 0)$ the following equality is satisfied

$$E\left(\int_0^\infty \phi_s \, dN_s\right) = E\left(\int_0^\infty \phi_s \lambda_s ds\right) .$$

Therefore $(M_t = N_t - \Lambda_t,\ t \geq 0)$ is an $\mathbf{F}$-local martingale.

If ϕ is a predictable process such that $\forall t$, $E(\int_0^t |\phi_s| \lambda_s ds) < \infty$, then $(\int_0^t \phi_s dM_s,\ t \geq 0)$ is an $\mathbf{F}$-martingale.

An inhomogeneous Poisson process N with stochastic intensity λ_t can be viewed as time changed of $\tilde{N}$, a standard Poisson process $N_t = \tilde{N}_{\Lambda_t}$.

4.2 Itô's formula

The formula obtained in Section 3.4 generalizes to inhomogeneous Poisson process with stochastic intensity.

4.3 Exponential Martingales

Proposition 4.2 *Let N be an inhomogeneous Poisson process with stochastic intensity $(\lambda_t,\, t \geq 0)$, and $(\mu_t,\, t \geq 0)$ a predictable process such that $\int_0^t |\mu_s|\lambda_s\, ds < \infty$. Then, the process L defined by*

$$L_t = \begin{cases} \exp(-\int_0^t \mu_s\lambda_s\, ds) & \text{if} \quad t < T_1 \\ \prod_{n, T_n \leq t}(1+\mu_{T_n}) \exp(-\int_0^t \mu_s\lambda_s\, ds) & \text{if} \quad t \geq T_1 \end{cases} \tag{4.1}$$

is a local martingale, solution of

$$dL_t = L_{t-}\mu_t dM_t, \quad L_0 = 1\,. \tag{4.2}$$

Moreover, if μ is such that $\forall s, \mu_s > -1$,

$$L_t = \exp\left[-\int_0^t \mu_s\lambda_s ds + \int_0^t \ln(1+\mu_s)\, dN_s\right]\,.$$

The local martingale L is denoted by $\mathcal{E}(\mu{\star}M)$ and named the Doléans-Dade exponential of the process $\mu{\star}M$.

This process can also be written

$$L_t = \prod_{0<s\leq t} (1+\mu_s \Delta N_s\,)\exp\left[-\int_0^t \mu_s\,\lambda_s\,ds\right]\;.$$

Moreover, if $\forall t, \mu_t > -1$, then L is a non-negative local martingale, therefore it is a supermartingale and

$$\begin{aligned} L_t &= \exp\left[-\int_0^t \mu_s\lambda_s ds + \sum_{s\leq t}\ln(1+\mu_s)\Delta N_s\right] \\ &= \exp\left[-\int_0^t \mu_s\lambda_s ds + \int_0^t \ln(1+\mu_s)\,dN_s\right] \\ &= \exp\left[\int_0^t [\ln(1+\mu_s)-\mu_s]\,\lambda_s\,ds + \int_0^t \ln(1+\mu_s)\,dM_s\right]\;. \end{aligned}$$

The process L is a martingale if $\forall t$, $E(L_t) = 1$.

If μ is not greater than -1, then the process L defined in (4.1) is still a local martingale which satisfies $dL_t = L_{t-}\mu_t dM_t$. However it may be negative.

4.4 Change of Probability

Let μ be a predictable process such that $\mu > -1$ and $\int_0^t \lambda_s |\mu_s| ds < \infty$.

Let L be the positive exponential local martingale solution of

$$dL_t = L_{t-} \mu_t dM_t \,.$$

Assume that L is a martingale and let Q be the probability measure equivalent to P defined on $\mathcal{F}_t$ by $Q|_{\mathcal{F}_t} = L_t \, P|_{\mathcal{F}_t}$.

Under Q, the process

$$(M_t^\mu \stackrel{def}{=} M_t - \int_0^t \mu_s \lambda_s ds = N_t - \int_0^t (\mu_s + 1) \lambda_s \, ds \;, t \geq 0)$$

is a local martingale.

5 Compound Poisson Processes

5.1 Definition and Properties

Let λ be a positive number and $F(dy)$ be a probability law on $I\!R$. A (λ, F) compound Poisson process is a process $X = (X_t, t \geq 0)$ of the form

$$X_t = \sum_{k=1}^{N_t} Y_k$$

where N is a Poisson process with intensity $\lambda > 0$ and the $(Y_k, k \in I\!N)$ are i.i.d. square integrable random variables with law $F(dy) = P(Y_1 \in dy)$, independent of N.

Proposition 5.1 *A compound Poisson process has stationary and independent increments; the cumulative function of X_t is*

$$P(X_t \leq x) = e^{-\lambda t} \sum_n \frac{(\lambda t)^n}{n!} F^{*n}(x) \, .$$

If $E(|Y_1|) < \infty$, the process $(Z_t = X_t - t\lambda E(Y_1), t \geq 0)$ is a martingale and in particular, $E(X_t) = \lambda t E(Y_1) = \lambda t \int y F(dy)$.

If $E(Y_1^2) < \infty$, the process $(Z_t^2 - t\lambda E(Y_1^2), t \geq 0)$ is a martingale and $Var\ (X_t) = \lambda t E(Y_1^2)$.

Introducing the random measure $\mu = \sum_n \delta_{T_n, Y_n}$ on $I\!R^+ \times I\!R$ and denoting by $(f * \mu)_t$ the integral $\int_0^t \int_{I\!R} f(x)\mu(\omega, ds, dx)$, we obtain that

$$M_t^f = (f * \mu)_t - t\nu(f) = \int_0^t \int_{I\!R} f(x)(\mu(\omega, ds, dx) - ds\nu(dx))$$

is a martingale.

Corollary 5.2 *Let X be a (λ, F) compound Poisson process independent of W. Let*

$$dS_t = S_{t-}(\mu dt + dX_t)\,.$$

Then,

$$S_t = S_0 e^{\mu t} \prod_{k=1}^{N_t} (1 + Y_k)$$

In particular, if $1 + Y_1 > 0, a.s.$

$$S_t = S_0 \exp(\mu t + \sum_{k=1}^{N_t} \ln(1 + Y_k))\,.$$

The process $(S_t e^{-rt}, t \geq 0)$ is a martingale if and only if $\mu + \lambda E(Y_1) = r$.

5.2 Martingales

We now denote by ν the measure $\nu(dy) = \lambda F(dy)$, a (λ, F) compound Poisson process will be called a (λ, ν) compound Poisson process.

Proposition 5.3 *If X is a (λ, ν) compound Poisson process, for any α such that $\int_{-\infty}^{\infty} e^{\alpha u} \nu(du) < \infty$, the process*

$$Z_t^{(\alpha)} = \exp\left(\alpha X_t + t\left(\int_{-\infty}^{\infty} (1 - e^{\alpha u})\nu(du)\right)\right)$$

is a martingale and

$$E(e^{\alpha X_t}) = \exp\left(-t\left(\int_{-\infty}^{\infty} (1 - e^{\alpha u})\nu(du)\right)\right).$$

Proposition 5.4 *Let X be a (λ, ν) compound Poisson process, and f a bounded Borel function. Then, the process*

$$\exp\left(\sum_{k=1}^{N_t} f(Y_k) + t\int_{-\infty}^{\infty}(1 - e^{f(x)})\nu(dx)\right)$$

is a martingale. In particular

$$E\left(\exp\left(\sum_{k=1}^{N_t} f(Y_k)\right)\right) = \exp\left(-t\int_{-\infty}^{\infty}(1 - e^{f(x)})\nu(dx)\right)$$

For any bounded Borel function f, we denote by $\nu(f) = \int_{-\infty}^{\infty} f(x)\nu(dx)$ the product $\lambda E(f(Y_1))$.

Proposition 5.5 *Let X be a (λ, ν) compound Poisson process. The process*

$$M_t^f = \sum_{s \leq t} f(\Delta X_s) 1\!\!1_{\{\Delta X_s \neq 0\}} - t\nu(f)$$

is a martingale. Conversely, suppose that X is a pure jump process and that there exists a finite positive measure σ such that

$$\sum_{s \leq t} f(\Delta X_s) 1\!\!1_{\{\Delta X_s \neq 0\}} - t\sigma(f)$$

is a martingale for any f, then X is a $(\sigma(1), \sigma)$ compound Poisson process.

5.3 Hitting Times

Let $X_t = bt + \sum_{k=1}^{N_t} Y_k$.

Assume that the support of F is included in $]-\infty, 0]$.

The process $(\exp(uX_t - t\psi(u)), t \geq 0)$ is a martingale, with

$$\psi(u) = bu - \lambda \int_{-\infty}^{0} (1 - e^{uy}) F(dy)\,.$$

Let $T_x = \inf\{t : X_t > x\}$. Since the process X has no positive jumps, $X_{T_x} = x$.

Hence $E(e^{uX_{t\wedge T_x} - (t\wedge T_x)\psi(u)}) = 1$ and when t goes to infinity, one obtains

$$E(e^{ux - T_x\psi(u)} \mathbb{1}_{\{T_x < \infty\}}) = 1\,.$$

If ψ admits an inverse $\psi^\sharp$, one gets if T_x is finite

$$E(e^{-\lambda T_x}) = e^{-x\psi^\sharp(\lambda)}\,.$$

Setting $Z_k = -Y_k$, the random variables Z_k can be interpreted as losses for insurance companies. The process $z + bt - \sum_{k=1}^{N_t} Z_k$ is called the Cramer-Lundberg risk process. The time $\tau = \inf\{t : X_t \leq 0\}$ is the bankruptcy time for the company.

One sided exponential law. If $F(dy) = \theta e^{\theta y} \mathbb{1}_{\{y<0\}} dy$, one obtains $\psi(u) = bu - \dfrac{\lambda u}{\theta + u}$, hence inverting ψ,

$$E(e^{-\kappa T_x} \mathbb{1}_{\{T_x<\infty\}}) = e^{-x\psi^\sharp(\kappa)} ,$$

with

$$\psi^\sharp(\kappa) = \frac{\lambda + \kappa - \theta b + \sqrt{(\lambda + \kappa - \theta b)^2 + 4\theta b}}{2b} .$$

5.4 Change of Measure

Let X be a (λ, ν) compound Poisson process, $\widetilde{\lambda} > 0$ and $\widetilde{F}$ a probability measure on $I\!R$, absolutely continuous w.r.t. F and $\widetilde{\nu}(dx) = \widetilde{\lambda}\widetilde{F}(dx)$. Let

$$L_t = \exp\left(t(\lambda - \widetilde{\lambda}) + \sum_{s \le t} \ln \left(\frac{d\widetilde{\nu}}{d\nu} \right) (\Delta X_s) \right) .$$

Set $dQ|_{\mathcal{F}_t} = L_t dP|_{\mathcal{F}_t}$.

Proposition 5.6 *Under Q, the process X is a $(\widetilde{\lambda}, \widetilde{\nu})$ compound Poisson process.*

6 An Elementary Model of Prices including Jumps

Suppose that S is a stochastic process with dynamics given by

$$dS_t = S_{t-}(b(t)dt + \phi(t)dM_t), \tag{6.1}$$

where M is the compensated compensated martingale associated with a Poisson process and where b, ϕ are deterministic continuous functions, such that $\phi > -1$. The solution of (6.1) is

$$\begin{aligned} S_t &= S_0 \exp\left[-\int_0^t \phi(s)\lambda(s)ds + \int_0^t b(s)ds\right] \prod_{s\le t}(1+\phi(s)\Delta N_s) \\ &= S_0 \exp\left[\int_0^t b(s)ds\right] \exp\left[\int_0^t \ln(1+\phi(s))dN_s - \int_0^t \phi(s)\lambda(s)ds\right]. \end{aligned}$$

Hence $S_t \exp\left(-\int_0^t b(s)ds\right)$ is a strictly positive martingale.

We denote by r the deterministic interest rate and by $R_t = \exp\left(-\int_0^t r(s)ds\right)$ the discounted factor.

Any strictly positive martingale L can be written as $dL_t = L_{t-}\mu_t dM_t$ with $\mu > -1$.

Let $(Y_t = R_t S_t L_t, t \geq 0)$. Itô's calculus yields to

$$\begin{aligned} dY_t &\cong Y_{t-}\left((b(t) - r(t))dt + \phi(t)\mu_t d[M]_t\right) \\ &\cong Y_{t-}\left(b(t) - r(t) + \phi(t)\mu_t\lambda(t)\right) dt\,. \end{aligned}$$

Hence, Y is a local martingale if and only if $\mu_t\lambda(t) = -\dfrac{b(t) - r(t)}{\phi(t)}$.

Assume that $\mu > -1$ and $Q|_{\mathcal{F}_t} = L_t\, P|_{\mathcal{F}_t}$. Under Q, N is a Poisson process with intensity $((\mu(s) + 1)\lambda(s), s \geq 0)$ and

$$dS_t = S_{t_-}(r(t)dt + \phi(t)dM_t^\mu)$$

where $(M^\mu(t) = N_t - \int_0^t (\mu(s) + 1)\lambda(s)\, ds\,,\ t \geq 0)$ is the compensated Q-martingale.

Hence Q is the unique equivalent martingale measure.

6.1 Price Process

Let

$$dS_t = (\alpha S_{t-} + \beta)\, dt + (\gamma S_{t-} + \delta) dX_t \tag{6.2}$$

where X is a (λ, ν) compound Poisson process. The solution of (6.2) is a Markov process with infinitesimal generator

$$\mathcal{L}(f)(x) = \int_{-\infty}^{+\infty} [f(x + \gamma xy + \delta y) - f(x)]\, \nu(dy) + (\alpha x + \beta) f'(x)\,.$$

Let S be the solution of (6.2). The process $e^{-rt} S_t$ is a martingale if and only if

$$\gamma \int y\nu(dy) + \alpha = r, \quad \delta \int y\nu(dy) + \beta = 0\,.$$

Let $\widetilde{F}$ be a probability measure absolutely continuous with respect to F and

$$L_t = \exp\left(t(\lambda - \widetilde{\lambda}) + \sum_{s \le t} \ln\left(\frac{\widetilde{\lambda}}{\lambda}\frac{d\widetilde{F}}{dF}\right)(\Delta X_s)\right) .$$

Let $dQ|_{\mathcal{F}_t} = L_t dP|_{\mathcal{F}_t}$. The process $(S_t e^{-rt}, t \ge 0)$ is a Q-martingale if and only if

$$\widetilde{\lambda}\gamma \int y\widetilde{F}(dy) + \alpha = r, \quad \widetilde{\lambda}\delta \int y\widetilde{F}(dy) + \beta = 0$$

Hence, there are an infinite number of e.m.m.. One can change the intensity of the Poisson process, and/or the law of the jumps.