Uncertainty Principles in Signal Processing

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Uncertainty principle in Signal Processing

Assume that $\phi$ is a signal (a function depending of the time, for instance). Doing its spectrum analysis means: take its Fourier transform

$$\hat{\phi}(\omega) := \int_{\mathbb{R}} e^{-2i\pi \omega t} \phi(t) \, dt.$$  

The signal is given by $\hat{\phi}$ in the frequency domain.

One can reconstruct $f$ from its “spectrum” (under appropriate assumptions):

$$\phi(t) = \int_{-\infty}^{\infty} \hat{\phi}(\omega) e^{2i\pi \omega t} \, d\omega.$$  

The Uncertainty principle: a signal cannot be localized both in time and frequency.
Uncertainty principle in Quantum Mechanics

The position and the momentum (velocity) of a sub-atomic particle cannot both be known with arbitrary precision. The more precisely one is known, the less precisely the other can be known.

To quote Heisenberg: Any use of the words “position” and “velocity” with an accuracy exceeding that given by the uncertainty equation is just as meaningless as the use of words whose sense is not defined.” (from Physical Principles of the Quantum Theory, 1930)

- it is impossible to determine simultaneously both the position and velocity of a particle with any great degree of accuracy or certainty.
- it is a statement about the nature of the system itself as described by the equations of quantum mechanics.
Let $f$ integrable in $\mathbb{R}^d$. Let its Fourier transform be

$$\hat{f}(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} \, dx$$

well defined for $f \in L^1(\mathbb{R}^d)$, then, using Plancherel Identity, for $f \in L^2(\mathbb{R}^d)$, then for tempered distributions.

**Uncertainty Principle.** A function $f$ and its Fourier transform $\hat{f}$ cannot be simultaneously well localized.

In quantum mechanics, the distribution of the position (resp. the momentum) of a particle is given by its wave function (resp. the Fourier transform of its wave function).
Uncertainty principle: examples of statements

Let $\delta_a$ be the Dirac mass at $a \in \mathbb{R}^d$, which is highly localized. Then its Fourier transform

$$
\hat{\delta}_a(\xi) := \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} \, d\delta_a(x) = e^{-ia\xi}
$$

has modulus 1 everywhere.

Let $f$ be compactly supported. Then its Fourier transform vanishes only at isolated points.

Proof: $\hat{f}$ extends into a holomorphic function on the complex plane.

Remark: one can exchange the role of $f$ and $\hat{f}$.

Hardy’s Theorem. If

$$
|f(x)| \leq Ce^{-a|x|^2} \quad \quad |\hat{f}(\xi)| \leq Ce^{-b|\xi|^2},
$$

then

- if $ab > 1/4$, the function $f$ is 0;
- if $ab = 1/4$, the function $f$ is, up to a constant, the Gaussian function $e^{-a|x|^2}$.
Heisenberg Uncertainty Inequality

Heisenberg Inequality (in Dimension one) is

$$2\Delta x \Delta \xi \geq 1.$$ 

Mathematically, this means the inequality

$$2 \left( \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \right)^{1/2} \times \left( \frac{1}{2\pi} \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} \geq \int_{\mathbb{R}} |f(x)|^2 \, dx,$$

with equality for Gaussian functions, that is, functions $$e^{-a|x-x_0|^2}$$.

$$\hat{f}(\xi) := \int_{-\infty}^{+\infty} e^{-ix\xi} f(x) \, dx$$

and Plancherel Identity reads

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\hat{f}(x)|^2 \, dx.$$
The proof! (for physicists)

Want to prove that

\[ 2 \left( \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \right)^{1/2} \times \left( \frac{1}{2\pi} \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} \geq 1, \]

for \( \int_{\mathbb{R}} |f(x)|^2 \, dx = 1. \)

Use of Plancherel Identity

\[ \frac{1}{2\pi} \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, d\xi = \int |\frac{\partial}{\partial x} f(x)|^2 \, dx. \]

\[ 1 = \int |f|^2 \, dx = -2 \int \left( \frac{\partial}{\partial x} f \right) (xf) \, dx. \]

Then use of Cauchy Schwarz Inequality.

Define the operators \( Q : \) multiplication by \( x, \) and \( P := -i \frac{\partial}{\partial x}. \)

Heisenberg commutation properties :

\[ [P, Q] := PQ - QP = -i \text{Id}. \]
The spectral gap of the Harmonic oscillator

\[ 2 \left( \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \right)^{1/2} \times \left( \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, d\xi \right)^{1/2} \geq \int_{\mathbb{R}} |f(x)|^2 \, dx \]

leads to

\[ \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx + \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, d\xi \geq \int_{\mathbb{R}} |f(x)|^2 \, dx, \]

or, which is equivalent,

\[ \left\langle x^2 f - \left( \frac{d}{dx} \right)^2 f, f \right\rangle \geq \|f\|_2^2. \]

1 is the smallest eigenvalue of the harmonic oscillator \( \mathcal{H} := x^2 - \left( \frac{d}{dx} \right)^2 \) and \( e^{-x^2/2} \) is the corresponding eigenfunction.
The eigenvalues of the Harmonic oscillator

$2n + 1$ is the $n$th eigenvalue of the harmonic oscillator $\mathcal{H} := x^2 - \left(\frac{d}{dx}\right)^2$ and the Hermite function $H_n(x)e^{-x^2/2}$ is the corresponding eigenfunction. Here $H_n$ is the polynomial

$$H_n(x) := (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n (e^{-x^2}).$$

Moreover, by the Courant-Fischer-Weyl min-max principle for compact operators (here the inverse of $\mathcal{H}$), then $2n + 1$ is equal to

$$\max \min \left\{ \langle Hf, f \rangle ; f \in (f_1, \cdots, f_n)^\perp, \|f\|_2 = 1 \right\}.$$

Extrema are given by Hermite functions.
Slepian Extremal problem

How large on $[-1, +1]$ are functions in $L^2(\mathbb{R})$ with spectrum in $[-c, +c]$? Let $B_c$ be the space of such functions.

Then

$$\lambda_0(c) = \max \left\{ \int_{-1}^{+1} |f|^2 dx ; \quad ||f||_2 = 1, f \in B_c \right\}.$$ 

Then

$$\lambda_n(c) = \min \max_{f_1, \cdots, f_n} \left\{ \int_{-1}^{+1} |f|^2 dx ; \quad f \in (f_1, \cdots, f_n) \perp, ||f||_2 = 1, f \in B_c \right\}.$$ 

Consider the operator on $L^2([-1, +1])$

$$\mathcal{F}_c(f)(x) := \left( \frac{c}{2\pi} \right)^{1/2} \int_{-1}^{+1} e^{icx\xi} f(\xi) d\xi.$$ 

Then $\lambda_n(c)$ are the eigenvalues of $\mathcal{F}_c^* \mathcal{F}_c$.

Key point: $\mathcal{F}_c^* \mathcal{F}_c(f) = f \ast \chi_{[-c, +c]}$ is compact.
The asymptotic behavior of $\lambda_n(c)$.

(Landau and Widom 1980)
Asymptotically for $c$ tending to $\infty$, the sequence $\lambda_n(c)$ stays close to 1 for $n \leq \frac{2}{\pi} c$, then decreases exponentially rapidly.

For $c$ large, and looking at the restrictions of $B_c$ on the interval $[-1, +1]$, roughly speaking one may do as if this space was of finite dimension $N \approx \frac{2}{\pi} c$.

May be extremely useful for compression of data in Signal Processing.
The eigenfunctions.

Let $\psi_{n,c}$ be the eigenfunction corresponding to $\lambda_{n,c}$.

The main tool: (Slepian) The eigenfunctions $\lambda_n$ are also eigenfunctions of an explicit Sturm-Liouville operator on $(-1, +1)$

$$\mathcal{L}_c \phi := -\frac{d}{dx} [(1 - x^2)\phi'] + c^2 x^2 \phi.$$

Arises in finding particular “radial” solutions of the Helmotz equation $\Delta \Phi + k^2 \Phi = 0$ in three dimensions in the “prolate spheroidal coordinates”. Explains the name: Prolate Spheroidal Wave Functions PSWF.

C. Niven: On the Conduction of Heat in Ellipsoids of Revolution, Philosophical Trans. R. Soc. Lond. 1880 171, 117-151. In this remarkable piece of work, Niven has developed a detailed computational and asymptotic methods for the PSWFs and the eigenvalues $\chi_n(c)$. Justifies to see PSWF appear in many scientific fields.