Generalized Nash Equilibrium Problem: existence, uniqueness and reformulations

Didier Aussel

Univ. de Perpignan, France

CIMPA-UNESCO school, Delhi
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Outline of the 7 lectures

- **Generalized Nash Equilibrium**
  a- Reformulations
  b- Existence of equilibrium

- **Variational inequalities: motivations, definitions**
  a- Motivations, definitions
  b- Existence of solutions
  c- Stability
  d- Uniqueness

- **Quasiconvex optimization**
  a- Classical subdifferential approach
  b- Normal approach

- Whenever a set-valued map is not really set-valued...

- **Quasivariational inequalities**
Variational inequalities: basic facts and existence

I- Some examples (motivation)

II- First Existence results
   a- The linear case
   b- The finite dimensional case

III- A general case
Notations

- $X$ a topological vector space
- $X^*$ its topological dual ($w^*$-top.)
- $\langle \cdot, \cdot \rangle$ the duality product

Stampacchia variational inequality:

Let $T : X \to 2^{X^*}$ be a map and $C$ be a nonempty subset of $X$. Find $\bar{x} \in C$ such that there exists $\bar{x}^* \in T(\bar{x})$ for which $\langle \bar{x}^*, y - \bar{x} \rangle \geq 0, \forall y \in C$.

Notation:

$S(T, C)$ set of solutions ($\subset C$).
Notations

- $X$ a topological vector space
- $X^*$ its topological dual ($w^*$-top.)
- $\langle \cdot, \cdot \rangle$ the duality product

**Stampacchia variational inequality:**
Let $T : X \to 2^{X^*}$ be a map and $C$ be a nonempty subset of $X$.

Find $\bar{x} \in C$ such that there exists $\bar{x}^* \in T(\bar{x})$ for which

$$\langle \bar{x}^*, y - \bar{x} \rangle \geq 0, \quad \forall y \in C.$$ 

Notation: $S(T, C)$ set of solutions ($\subset C$).
Let $f : X \to \mathbb{IR} \cup \{+\infty\}$ and $C \subseteq \text{dom } f$ be a convex subset.

\[
(P) \quad \text{find } \bar{x} \in C : f(\bar{x}) = \inf_{x \in C} f(x)
\]

**Necessary condition:** $f$ is lsc + ...

if $\bar{x}$ is a solution of $(P)$ then

\[
\bar{x} \in S(\partial f(\bar{x}), C).
\]
Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $C \subseteq \text{dom } f$ be a convex subset.

(\(P\)) find $\bar{x} \in C : f(\bar{x}) = \inf_{x \in C} f(x)$

**Necessary condition:** $f$ is lsc + ...

if $\bar{x}$ is a solution of $\text{(P)}$ then

$$\bar{x} \in S(\partial f(\bar{x}), C).$$

**Perfect case:** $f$ convex

$f : X \to \mathbb{R} \cup \{+\infty\}$ a proper convex function

$C$ a nonempty convex subset of $X$, $\bar{x} \in C + C.Q.$

Then

$$f(\bar{x}) = \inf_{x \in C} f(x) \iff \bar{x} \in S(\partial f, C)$$
Let $f : X \to \mathbb{IR} \cup \{+\infty\}$ and $C \subseteq \text{dom } f$ be a convex subset.

\[ (P) \quad \text{find } \bar{x} \in C : f(\bar{x}) = \inf_{x \in C} f(x) \]

**Necessary condition:** $f$ is lsc + ...

if $\bar{x}$ is a solution of $(P)$ then

\[ \bar{x} \in S(\partial f(\bar{x}), C). \]

**Perfect case:** $f$ convex

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Then

\[ f(\bar{x}) = \inf_{x \in C} f(x) \iff \bar{x} \in S(\partial f, C) \]

**What about quasiconvex case?** see in a future lecture
Another motivation

Signorini’s frictionless contact problem

\[ \omega \]

\[ \Gamma_1 \]

\[ \Gamma_2 \]

\[ \Gamma_3 \]

surface traction \( f_2 \in [L^2(\Gamma_2)]^k \)

volume force \( f_0 \in [L^2(\Omega)]^k \)

Frictionless contact

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Generalized Nash Equilibrium Problem: existence, uniqueness and
**Notations**

**Functional spaces:**

\[ S^k = \{ \sigma = (\sigma_{ij})_{ij} \in \mathbb{R}^{k \times k} : \sigma_{ij} = \sigma_{ji} \} = \mathbb{R}^{k \times k} \]

\[ W = \{ v \in H^1(\Omega)^k : v = 0 \text{ sur } \Gamma_1 \} \]

\[ Q = \{ q = (q_{ij}) \in L^2(\Omega)^{k \times k} : q_{ij} = q_{ji}, \quad 1 \leq i, j \leq k \} = L^2(\Omega)^{k \times k} \]

\[ W_2 = \{ v \in W : v_{ij} \leq 0 \text{ a.e. on } \Gamma_3 \} \]

**Deformation operator:** \( \varepsilon : H^1(\Omega)^k \rightarrow Q \)

\[
\varepsilon_{ij}(u) = \frac{1}{2} \left[ \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right], \quad 1 \leq i, j \leq k.
\]

**Scalar product:** \( \langle p, q \rangle_Q = \int_\Omega p_{ij}(x)q_{ij}(x) \, dx \) et \( \langle u, v \rangle_W = \langle \varepsilon(u), \varepsilon(v) \rangle_Q \)

**Elasticity operator:** \( F : \Omega \times S^k \rightarrow S^k \)

**Stress function:** \( \sigma : H^1(\Omega)^k \rightarrow Q \) defined by

\[
\sigma(u) : \begin{cases} 
\Omega & \rightarrow S^k \\
x & \mapsto F(x, \varepsilon(u)(x)).
\end{cases}
\]
Formulations of the problem:

Find a displacement field $u : \Omega \to \mathbb{R}$ such that

$$
\begin{align*}
-\text{Div } \sigma(u) &= f_0 & \text{on } \Omega \\
u &= 0 & \text{on } \Gamma_1 \\
\sigma(u)\nu &= f_2 & \text{on } \Gamma_2 \\
u\nu &\leq 0, \; \sigma(u)\nu &\leq 0, \; \sigma(u)\nu u\nu &= 0, \; \sigma(u)\tau &= 0 & \text{on } \Gamma_3
\end{align*}
$$
Formulations of the problem:

Find a displacement field $u : \Omega \rightarrow \mathbb{R}$ such that

\[-\text{Div} \, \sigma(u) = f_0 \quad \text{on} \quad \Omega\]
\[u = 0 \quad \text{on} \quad \Gamma_1\]
\[\sigma(u)\nu = f_2 \quad \text{on} \quad \Gamma_2\]
\[u\nu \leq 0, \quad \sigma(u)\nu \leq 0, \quad \sigma(u)\nu u\nu = 0, \quad \sigma(u)\tau = 0 \quad \text{on} \quad \Gamma_3\]

Variational formulation:

Find $u \in W_2$ such that

\[\langle \sigma(u), \varepsilon(v) - \varepsilon(u) \rangle_Q \geq \langle f, v - u \rangle_{W}, \quad \forall \, v \in W_2\]

where $f$ is an element of $W$ defined by

\[\langle f, v \rangle_W = \int_{\Omega} f_0 \cdot v \, dx + \int_{\Gamma_2} f_2 \cdot v \, da.\]
Let $C \subseteq X$ and $T : C \to 2^{X^*}$ be a map.
We denote by

- $S_w(T, C)$ set of weak solutions of Stampacchia V.I.

$$\text{Find } \bar{x} \in C \text{ such that } \forall \ y \in C, \ \exists \bar{x}^* \in T(\bar{x}) : \langle \bar{x}^*, y - \bar{x} \rangle \geq 0$$

- $M(T, C)$ set of solutions of the Minty V.I.

$$\text{Find } \bar{x} \in C \text{ such that } \forall \ y \in C \text{ and } \forall \ y^* \in T(y) : \langle y^*, y - \bar{x} \rangle \geq 0.$$ 

- $LM(T, C)$ set of local solutions of Minty V.I., i.e. $\bar{x} \in LM(T, C)$ if it exists a neighb. $V$ of $\bar{x}$ such that $\bar{x} \in M(T, V \cap C)$.

If $T$ is pseudomonotone, then $LM(T, C) = M(T, C)$. 
II - First Existence results

a- The linear case
b- The finite dimension case
c- V.I. under monotonicity
d- Other assumptions
Let $H$ be an Hilbert space, $f$ an element of $H^* = H$ and $a : H \times H \to H$ a bilinear application.

**Theorem (Stampacchia 64)**

Let $C$ be a nonempty closed convex subset of $H$. If $a$ is continuous and satisfies the following property

$$\exists \alpha > 0 \text{ such that } a(v, v) \geq \alpha \|v\|^2, \quad \forall v \in H,$$

then

there exists $\bar{x} \in C$ such that $a(\bar{x}, y - \bar{x}) \geq \langle f, y - \bar{x} \rangle, \quad \forall y \in C.$

Theorem (Lax-Milgram)

If $a$ is continuous and satisfies the following property

$$\exists \alpha > 0 \text{ such that } a(v, v) \geq \alpha \|v\|^2, \; \forall v \in H,$$

then there exists a unique $\bar{x} \in H$ such that

$$a(\bar{x}, u) = \langle f, u \rangle, \; \forall u \in H.$$

Proof. From the Theorem of Stampacchia, there exists $u \in H$ such that

$$a(u, v - u) \geq \langle f, v - u \rangle, \; \forall v \in H$$

or in other words $\varphi_u(v) \geq \varphi_u(u), \; \forall v \in H$ where

$$\varphi_u(v) = a(u, v) - \langle f, v \rangle.$$

But $\varphi_u$ is linear and therefore $\varphi_u \equiv 0$. $\blacksquare$
Let $f : X \to X^*$ be a function and $C$ be a nonempty subset of $X$.

Find $\bar{x} \in C$ such that

$$\langle f(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \forall y \in C.$$ 

First property:

$$S(f, C) \cap \text{int}(C) = \{ x \in \text{int}(C) : f(x) = 0 \}$$
Proposition (Stampacchia 1966)

Let $C$ be a nonempty convex compact subset of $\mathbb{R}^n$ and $f : C \to \mathbb{R}^n$ a continuous function. Then $S(f, C)$ is nonempty, that is,

there exists $\bar{x} \in C$ such that $\langle f(\bar{x}), y - \bar{x} \rangle \geq 0$, $\forall y \in C$. 

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Generalized Nash Equilibrium Problem: existence, uniqueness and
Proposition (Stampacchia 1966)

Let $C$ be a nonempty convex compact subset of $\mathbb{R}^n$ and $f : C \rightarrow \mathbb{R}^n$ a continuous function. Then $S(f, C)$ is nonempty, that is,

there exists $\bar{x} \in C$ such that $\langle f(\bar{x}), y - \bar{x} \rangle \geq 0$, $\forall y \in C$.

Proof: Let us consider the application

$$
\psi : C \rightarrow C \quad x \mapsto \psi(x) = P_C \circ (Id - f)(x) = P_C(x - f(x))
$$

$\psi$ is continuous on the convex compact set $C$ and therefore, according to the Brouwer fixed point theorem, there exists a point $\bar{x} \in C$ such that

$$
\bar{x} = \psi(\bar{x}) \iff \bar{x} = P_C(\bar{x} - f(\bar{x}))
\iff \langle \bar{x} - f(\bar{x}) - \bar{x}, y - \bar{x} \rangle \leq 0, \forall y \in C
\iff \bar{x} \in S(f, C).
$$
The compactness hypothesis can be replaced by a "coercivity condition":

**Proposition**

Let $C$ be a nonempty closed convex subset of $\mathbb{R}^n$ and $f : C \rightarrow \mathbb{R}^n$ a continuous function. The following conditions are equivalent:

- $S(f, C)$ is nonempty.
- $\exists r > 0$ such that $S(f, C \cap \overline{B}(0, r)) \cap B(0, r) \neq \emptyset$, 

Where $S(f, C)$ denotes the set of equilibrium points of $f$ on $C$. 

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Generalized Nash Equilibrium Problem: existence, uniqueness and
The compactness hypothesis can be replaced by a "coercivity condition" in term of the function:

**Proposition**

Let $C$ be a nonempty closed convex subset of $\mathbb{R}^n$ and $f : C \rightarrow \mathbb{R}^n$ a continuous function. If $f$ is coercive, that is,

$$\exists x_0 \in C \text{ such that } \lim_{x \to x_0} \frac{\langle f(x) - f(x_0), x - x_0 \rangle}{\|x - x_0\|} = +\infty$$

then $S(f, C)$ is nonempty.
Theorem

Let $C$ be a nonempty convex compact subset of $\mathbb{R}^n$ and $F : C \to 2^{\mathbb{R}^n}$ a upper semicontinuous map with convex compact values. Then $S(F, C)$ is nonempty, that is,

there exist $\bar{x} \in C$ and $\bar{x}^* \in F(\bar{x})$

such that $\langle \bar{x}^*, y - \bar{x} \rangle \geq 0$, $\forall y \in C$. 

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Generalized Nash Equilibrium Problem: existence, uniqueness and
Theorem

Let $C$ be a nonempty convex compact subset of $\mathbb{R}^n$ and $F : C \to 2^{\mathbb{R}^n}$ a upper semicontinuous map with convex compact values. Then $S(F, C)$ is nonempty, that is,

there exist $\bar{x} \in C$ and $\bar{x}^* \in F(\bar{x})$

such that $\langle \bar{x}^*, y - \bar{x} \rangle \geq 0$, $\forall y \in C$.

can be found, e.g., Borwein, J. and Lewis, A., Convex Analysis and Nonlinear Optimization, 2000
Theorem

Let $C$ be a nonempty convex compact subset of $\mathbb{R}^n$ and $F : C \to 2^{\mathbb{R}^n}$ a upper semicontinuous map with convex compact values. Then $S(F, C)$ is nonempty, that is,

there exist $\bar{x} \in C$ and $\bar{x}^* \in F(\bar{x})$

such that $\langle \bar{x}^*, y - \bar{x} \rangle \geq 0, \ \forall \ y \in C.$

$F$ is upper semicontinuous at $x$ if

$\forall \ V$ neighb. of $F(x), \ \exists \ U$ neighb. of $x$ such that $F(U) \subset V.$
Proof. Since $F$ is cusco, its range $F(C)$ is compact and the map $T = (Id - F) \circ P_C$ is also cusco since $P_C$ is continuous (Exercise). Now one can observe that $T(co(C - F(C))) = co(C - F(C))$. Therefore $T$ has a fixed point (say $x_0$) on $C - F(C)$, that is

$$x_0 \in T(x_0) \iff x_0 \in (Id - F) \circ P_C(x_0)$$

$$\iff \begin{cases} \bar{x} = P_C(x_0) \\ \bar{x}^* = \bar{x} - x_0 \in F(\bar{x}) \end{cases}$$

$$\iff \begin{cases} \langle x_0 - \bar{x}, y - \bar{x} \rangle \leq 0, \quad \forall y \in C \\ \bar{x}^* = \bar{x} - x_0 \in F(\bar{x}) \end{cases}$$

and thus $\bar{x} \in S(F, C)$. □
Actually the above existence theorem 6 is equivalent to the Kakutani Fixed point theorem. Indeed, let $C$ be a nonempty convex compact subset of $\mathbb{IR}^n$ and $T : C \rightarrow 2^C$ be a cusco map. Observing that the set-valued map $F = \text{Id} - T$ is also cusco, Theorem 6 implies the solvability of the set-valued Stampacchia variational inequality defined by $F$ and $C$, that is there exists $\bar{x} \in C$ such that

$$\bar{x} \in S(\text{Id} - T, C) \iff \begin{cases} \exists u^* \in (\text{Id} - T)(\bar{x}) \text{ such that} \\ \langle u^*, y - \bar{x} \rangle \geq 0, \forall y \in C \end{cases}$$

$$\iff \begin{cases} \exists \bar{x}^* \in T(\bar{x}) \text{ such that} \\ \langle \bar{x} - \bar{x}^*, y - \bar{x} \rangle \geq 0, \forall y \in C \end{cases}$$

Taking $y = \bar{x}^*$, one has

$$\langle \bar{x} - \bar{x}^*, \bar{x}^* - \bar{x} \rangle \geq 0$$

showing that $\bar{x}$ is a fixed point of $T$. \[\square\]
II

Existence of solution

c - V.I. under monotonicity
Theorem (1966)

Let $X$ be a reflexive Banach space, $C$ be a weakly compact convex nonempty subset of $X$. If $f : C \rightarrow X^*$ is a monotone function, continuous on finite dimension subsets, then $S(f, C)$ is nonempty.

Hartmann, G. and Stampacchia, G., Acta Mathematica, 115 (1966), 271-310
### Theorem (1966)

Let $X$ be a reflexive Banach space, $C$ be a weakly compact convex nonempty subset of $X$. If $f : C \to X^*$ is a monotone function, continuous on finite dimension subsets, then $S(f, C)$ is nonempty.

### Theorem

Let $X$ be a reflexive Banach space, $C$ be a nonempty convex and weakly compact subset of $X$. If $F : C \to 2^{X^*}$ is a monotone map with convex weakly compact values and continuous on finite dimension subsets, then $S(F, C)$ is nonempty.

can be found, e.g., in Shih, M.H. and Tan, K.K., JMAA, 134 (1988), 431-440
Monotonicities

- $T$ is monotone iff $\forall x, y \in X, \forall x^* \in T(x)$ and $\forall y^* \in T(x)$
  \[ \langle y^* - x^*, y - x \rangle \geq 0. \]

- $T$ is pseudomonotone iff
  $\exists x^* \in T(x) : \langle x^*, y - x \rangle \geq 0 \Rightarrow \langle y^*, y - x \rangle \geq 0, \forall y^* \in T(y)$.

- $T$ is quasimonotone iff
  $\exists x^* \in T(x) : \langle x^*, y - x \rangle > 0 \Rightarrow \langle y^*, y - x \rangle \geq 0, \forall y^* \in T(y)$.

\begin{center}
  \begin{tabular}{c}
  monotone \\
  $\Downarrow$ \\
  pseudomonotone \\
  $\Downarrow$ \\
  quasimonotone
  \end{tabular}
\end{center}
Weakening monotonicity: why?

Let \( f : X \to \mathbb{R} \) be differentiable

- \( f \) is convex iff \( df \) is monotone
  \[ df(x)(y - x) > 0 \Rightarrow df(y)(y - x) \geq 0 \]
- \( f \) is pseudoconvex iff \( df \) is pseudomonotone
  \[ df(x)(y - x) > 0 \Rightarrow df(y)(y - x) \geq 0 \]
- \( f \) is quasiconvex iff \( df \) is quasimonotone
  \[ df(x)(y - x) > 0 \Rightarrow df(y)(y - x) \geq 0 \]

Let \( f : X \to \mathbb{R} \cup \{+\infty\} \) be lsc

- \( f \) is convex iff \( \partial f \) is monotone
  \[ \forall x^* \in \partial f(x), y^* \in \partial f(y), \langle y^* - x^*, y - x \rangle \geq 0 \]
- \( f \) is pseudoconvex iff \( \partial f \) is pseudomonotone
  \[ \exists x^* \in \partial f(x) : \langle x^*, y - x \rangle \geq 0 \]
  \[ \Rightarrow \forall y^* \in \partial f(y), \langle y^*, y - x \rangle \geq 0 \]
- \( f \) is quasiconvex iff \( \partial f \) is quasimonotone
  \[ \exists x^* \in \partial f(x) : \langle x^*, y - x \rangle > 0 \]
  \[ \Rightarrow \forall y^* \in \partial f(y), \langle y^*, y - x \rangle \geq 0 \]
Some previous attempts:

- J.C. Yao (1994) with pseudomonotone map on reflexive space
- N. Hadjisavvas and S. Schaible (1996) with quasimonotone + existence of inner points
- Dinh The Luc (2001) with singlevalued densely pseudomonotone maps
**Theorem**

* C nonempty convex compact subset of $X$.
* $T : C \rightarrow 2^{X^*}$ quasimonotone
  + upper hemi-continuous
  + $T(x) \neq \emptyset$ convex $w^*$-compact.

Then $S(T, C) \neq \emptyset$.

$T$ is upper hemicontinuous on $X$ if the restriction of $T$ to any line segment is usc with respect to the $w^*$-topology.
**Theorem**

* C nonempty closed convex subset of *X*.
* \( \varphi : C \times C \to \mathbb{R} \) KKM-application
  + \( \varphi(x, .) \) usc quasiconcave, \( \forall x \)
  + \( \exists \tilde{x} \in C \) with \( \{ y \in C : \varphi(\tilde{x}, y) \geq 0 \} \) compact.

Then \( \exists \bar{y} \in C \) such that \( \varphi(x, \bar{y}) \geq 0, \forall x \in C. \)
Theorem

C nonempty convex compact subset of X.
T : C → 2^{X*} quasimonotone
  + upper hemi-continuous
  + T(x) ≠ ∅ convex w*-compact.

Then S(T, C) ≠ ∅.

Theorem

C nonempty closed convex subset of X.
ϕ : C × C → IR KKM-application
  + ϕ(x, .) usc quasiconcave, ∀ x
  + ∃ TIMER{y ∈ C : ϕ TIMER{x, y} ≥ 0} compact.

Then ∃ TIMER{y ∈ C such that ϕ(x, TIMER{y}) ≥ 0, ∀ x ∈ C.
Upper sign-continuity

- \( T : X \to 2^{X^*} \) is said to be *upper sign-continuous* on \( C \) iff for any \( x, y \in C \), one have:

\[
\forall t \in ]0, 1[, \quad \inf_{x^* \in T(x_t)} \langle x^*, y - x \rangle \geq 0
\]

\[
\implies \quad \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0
\]

*where* \( x_t = (1 - t)x + ty \).
• $T : X \rightarrow 2^{X^*}$ is said to be upper sign-continuous on $C$ iff for any $x, y \in C$, one have:

\[
\forall t \in ]0, 1[, \quad \inf_{x^* \in T(x_t)} \langle x^*, y - x \rangle \geq 0
\]

\[\implies \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0\]

where $x_t = (1 - t)x + ty$.

upper semi-continuous
\[\Downarrow\]
upper hemi-continuous
\[\Downarrow\]
upper sign-continuous
Definition

Let $T : C \rightarrow 2^{X^*}$ be a set-valued map.

$T$ is called locally upper sign-continuous on $C$ if, for any $x \in C$ there exist a neighb. $V_x$ of $x$ and a upper sign-continuous set-valued map $\Phi_x : V_x \rightarrow 2^{X^*}$ with nonempty convex $w^*$-compact values such that $\Phi_x(y) \subseteq T(y) \setminus \{0\}, \forall y \in V_x$

upper semi-continuous
\[\downarrow\]
upper hemicontinuous
\[\downarrow\]
upper sign-continuous
\[\downarrow (\ast)\]
locally upper sign-continuous
**Proposition**

$C$ nonempty convex subset of $X$ and $T : C \rightarrow 2^{X^*}$ a map.

i) If $T$ is pseudomonotone, then $\text{LM}(T, C) = \text{M}(T, C)$.

ii) If $T$ is locally upper sign-continuous at $x$ then $\text{LM}(T, C) \subseteq \text{S}_w(T, C)$.

iii) If, additionally to ii), the maps $S_x$ are convex valued, then $\text{LM}(T, C) \subseteq \text{S}_w(T, C) = \text{S}(T, C)$. 

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Theorem

$C$ nonempty convex compact subset of $X$.

$T : C \rightarrow 2^{X^*}$ quasimonotone
+ $T(x) \neq \emptyset$ convex w*-compact
+ upper hemicontinuous

Then $S(T, C) \neq \emptyset$. 
Theorem

$C$ nonempty convex compact subset of $X$.

$T : C \to 2^{X^*}$ quasimonotone

$+ upper sign-continuous$

Then $S(T, C) \neq \emptyset$. 

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Theorem

C nonempty convex compact subset of X.

\[ T : C \rightarrow 2^{X^*} \text{ quasimonotone} \]
\[ \quad + \ T(x) \neq \emptyset \text{ convex w*-compact} \]
\[ \quad + \text{ locally upper sign-continuous on } C \]

Then \[ S(T, C) \neq \emptyset. \]
Theorem

Let $C$ be a nonempty convex subset of $X$. Let $T : C \to 2^{X^*}$ be quasimonotone and locally upper sign-continuous on $C$.

Let $\rho > 0$, $\forall x \in C \setminus \overline{B}(0, \rho)$, $\exists y \in C$ with $\|y\| < \|x\|$ such that $\forall x^* \in T(x)$, $\langle x^*, x - y \rangle \geq 0$.

And there exists $\rho' > \rho$ such that $C \cap \overline{B}(0, \rho')$ is weakly compact $\neq \emptyset$.

Then $S(T, C) \neq \emptyset$. 

Existence: complete form
Hartman and Stampacchia: sum of monotone operator and continuous function

J.C. Yao, N.D. Yen, B.T. Kien and many others: continuous maps with degre technics


