Generalized Nash Equilibrium Problem: existence, uniqueness and reformulations

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Outline of the 7 lectures

- **Generalized Nash Equilibrium**
  - a- Reformulations
  - b- Existence of equilibrium

- **Variational inequalities: motivations, definitions**
  - a- Motivations, definitions
  - b- Existence of solutions
  - c- Stability
  - d- Uniqueness

- **Quasiconvex optimization**
  - a- Classical subdifferential approach
  - b- Normal approach

- **Whenever a set-valued map is not really set-valued...**

- **Quasivariational inequalities**
Outline of the talk:

0- Basic data on electricity markets
I- Notations and definitions: Nash problems
II- A simple day-ahead market model
III- Variational reformulations of GNEP
0- The European energy markets

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Generalized Nash Equilibrium Problem: existence, uniqueness and
This EPEX market is open to any country and concerns Austria, France, Germany and Switzerland:
The French part of the market

Power  | Spot Market

Day-Ahead Auction | France

Day-Ahead Auction with Delivery in the French TSO Zone

Contract size  
The minimum volume amounts to 1 MW for individual hours and blocks.

Prices in bid  
The minimum price change is EUR 0.01 per MWh.

Underlying  
Electricity traded for delivery on the following day in 24 one-hour intervals.

Place of delivery  
Deliveries are made within the French transmission system managed by RTE.

Auction hours  
The daily auction takes place at 12:00 am, seven days a week, year-round, including statutory holidays, and is embedded in the “CWE Price Market Coupling”, which is itself linked to the Nordic Market through “Interim Tight Volume Coupling”.

Time of publication  
The results of the auction are made available from 12:40 am (CET).

Bid types

Individual hours  
Hourly bids comprise up to 256 price/quantity combinations for every hour of the following day. Prices must be between the technical price limits of EUR –3,000 per MWh and EUR 3,000 per MWh. The 256 prices in addition to the price ceiling and the price floor are not necessarily the same for every hour. A volume – whether positive, negative or nil – must be entered for the price ceiling and price floor. A price-inelastic bid is sent by entering the same quantity for the price ceiling and price floor.

Blocks  
Block bids are used to link several hours on an all-or-none basis, which means that either the bid is matched for all of the hours or it is rejected in its entirety.

Various standard blocks are predefined in order to facilitate entering bids. Moreover, the trading participants also have the possibility of entering user-defined blocks combining several random hours of their choice in addition. Block bids must comprise at least two hours of the day, which do not necessarily have to be consecutive hours.

The maximum volume for a block bid is 200 MW and a maximum of 40 block bids per portfolio can be entered.
A Nash equilibrium problem is a noncooperative game in which the decision function (cost/benefit) of each player depends on the decision of the other players.
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Denote by $N$ the number of players and each player $i$ controls variables $x^i \in \mathbb{R}^{n_i}$. The “total strategy vector” is $x$ which will be often denoted by

$$x = (x^i, x^{-i}).$$

where $x^{-i}$ is the strategy vector of the other players.
The strategy of player $i$ belongs to a strategy set $x^i \in X_i$.
Nash Equilibrium Problem (NEP)

- The strategy of player $i$ belongs to a strategy set
  \[ x^i \in X_i \]

- Given the strategies $x^{-i}$ of the other players, the aim of player $i$ is to choose a strategy $x^i$ solving

\[
P_i(x^{-i}) \quad \max \quad \theta_i(x^i, x^{-i})
\]
\[\text{s.t.} \quad x^i \in X_i\]

where $\theta_i(\cdot, x^{-i}) : \mathbb{R}^{n_i} \rightarrow \mathbb{R}$ is the decision function for player $i$. 

A vector $\bar{x}$ is a Nash Equilibrium if for any $i$, $\bar{x}^i$ solves $P_i(\bar{x}^{-i})$. 

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Generalized Nash Equilibrium Problem: existence, uniqueness and
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II

A simple day-ahead market model

In this electricity market model, there are three main actors: the producers, the consumers and the ISO. The market is represented by a set of nodes, denoted by $\mathcal{N} = \{1, \ldots, N\}$, and a set of lines, denoted by $\mathcal{L}$. Each node can be producer and/or consumer. The market is centralized by an Independant System Operator (ISO) which allocates the production of each producer in order to minimize the global cost of production.

Notations

- $A_i q_i + B_i q_i^2$ is the real cost of generation of the node $i$, $i = 1, \ldots, n$.
- $A_i (-q_i) - B_i (-q_i^2)$ is the utility for consumption of the node $i$, $i = n + 1, 2n$. 

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- Each consumer bids at the ISO his utility function.
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**The Parameters**

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- $A_i(-q_i) - B_i(-q_i^2)$ is the utility for consumption of the node $i$, $i = n + 1, 2n$. 
Why quadratic cost?

- In fact, in many markets, the bids are done by “boxes functions” or “step functions”
- But the bid offer has an increasing shape!

⇒ a quadratic approximation makes sense.
Knowing $a_{-i} = (a_j)_{j \neq i}$, the node $i$ computes $a_i, (q_j)_{j \in \mathcal{N}}$ in order to solve the following optimization problem:

\[
(P_i) \quad \max_{a_i, b_i, q} (a_i + 2B_i q_i)q_i - (A_i q_i + B_i q_i^2)
\]

such that

\[
\begin{align*}
A_i &\leq a_i \leq \bar{A}_i \\
(q_j)_{j \in \mathcal{N}} &\in Q(a)
\end{align*}
\]

with $0 \leq A_i \leq \bar{A}_i$. 
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(q_j)_{j\in \mathcal{N}} \in Q(a)
\end{cases}$$

with $0 \leq A_i \leq \bar{A}_i$.

The set $Q(a)$ denotes the set of the solutions of the ISO’s problem which is:

$$\min_q \quad \sum_{i\in \mathcal{N}} (a_i q_i + B_i q_i^2)$$

such that

$$\begin{cases}
q_i \geq 0, \ \forall i \in \mathcal{N} \\
q_1 + \cdots + q_{2n} = 0 \ (\lambda)
\end{cases}$$
The producer’s problem

Knowing \( a_{-i} = (a_j)_{j \neq i} \), the node \( i \) computes \( a_i, (q_j)_{j \in \mathcal{N}} \) in order to solve the following optimization problem:

Optimistic way

\[
(P_i) \quad \max_{a_i, b_i} \max_q (a_i + 2b_i q_i)q_i - (A_i q_i + B_i q_i^2) \\
\text{such that} \quad \begin{cases} 
A_i \leq a_i \leq \bar{A}_i \\
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with \( 0 \leq A_i \leq \bar{A}_i \).

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The producer’s problem

Knowing $a_{-i} = (a_j)_{j \neq i}$, the node $i$ computes $a_i, (q_j)_{j \in \mathcal{N}}$ in order to solve the following optimization problem:

Pessimistic way

\[
(P_i) \quad \max_{a_i, b_i} \min_{q} \quad (a_i + 2b_i q_i) q_i - (A_i q_i + B_i q_i^2)
\]

such that

\[
\left\{ \begin{array}{l}
A_i \leq a_i \leq \bar{A}_i \\
(q_j)_{j \in \mathcal{N}} \in Q(a)
\end{array} \right.
\]

with $0 \leq A_i \leq \bar{A}_i$.

The set $Q(a)$ denotes the set of the solutions of the \textbf{ISO’s problem} which is:

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such that

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q_i \geq 0, \; \forall i \in \mathcal{N} \\
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\]

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Generalized Nash Equilibrium Problem: existence, uniqueness and
Existence

Theorem

If $B_i > 0$, for any $i$, then the ISO’s problem $(Q(a))$ admits a unique solution.
Existence

**Theorem**

If $B_i > 0$, for any $i$, then the ISO's problem $(Q(a))$ admits a unique solution.

**Theorem**

If, for any possible bid $a$, the ISO's response $q^*(a)$ is such that, $q_i^* > 0$, for any $i$, then the Generalized Nash game admits at least a solution.
Existence

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If $B_i > 0$, for any $i$, then the ISO’s problem ($Q(a)$) admits a unique solution.

**Theorem**

If, for any possible bid $a$, the ISO’s response $q^*(a)$ is such that, $q_i^* > 0$, for any $i$, then the Generalized Nash game admits at least a solution.

Observe that the marginal price $\lambda = a_i + 2B_iq_i^*$ is the same for any producer.
III

Variational reformulation
Suppose that for any $\nu$ and any $x^{-\nu} \in \mathbb{R}^{n^{-\nu}}$, function $\theta_\nu(\cdot, x^{-\nu})$ is continuously differentiable and convex and $X_\nu(x^{-\nu})$ is convex.
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Denoting by

$$X(x) = \prod_{\nu} X_{\nu}(x^{-\nu}), \quad \forall x \in \mathbb{R}^n$$

and

$$F(x) = (\nabla_{x^1} \theta_{1}(x), \ldots, \nabla_{x^N} \theta_{N}(x)) \quad \in \mathbb{R}^n$$

we have the reformulation

$$\bar{x} \text{ gene. Nash equil. } \iff \begin{cases} \bar{x} \in X(\bar{x}) \text{ and } \\ \langle F(\bar{x}), y - \bar{x} \rangle \geq 0, \quad \forall y \in X(\bar{x}) \end{cases}$$
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that is a quasi-variational inequality.
We suppose that the functions $\theta_i(\cdot, x^{-i})$ are convex. We denote

$$S(x) = \prod_{i=1}^{n} X_i(x^{-i})$$

and

$$F(x) = \prod_{i=1}^{n} \partial_{x^i} \theta_i(x^i, x^{-i})$$

Then

$\bar{x}$ is a GNEP $\iff$ if $\bar{x}$ solves QVI($F, S$).
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Then

\( \bar{x} \) is a GNEP \( \iff \) if \( \bar{x} \) solves QVI\( (F, S) \).

What about nonconvex loss functions?

**Proposition**

Suppose that for any $\nu$ and any $x^{-\nu} \in \mathbb{R}^{n-\nu}$, function $\theta_{\nu}(\cdot, x^{-\nu})$ is continuously differentiable and *pseudoconvex*. Assume the Rosen’s law: there exists a nonempty convex set $X$ of $\mathbb{R}^n$ such that,

$$\forall \nu, \quad X_{\nu}(x^{-\nu}) = \{x^\nu \in \mathbb{R}^{n^\nu} : (x^\nu, x^{-\nu}) \in X\}.$$  

Then any solution of $S(F, X)$ is a solution of the GNEP.

where

$$F(x) = (\nabla_{x^1}\theta_1(x), \ldots, \nabla_{x^N}\theta_N(x)) \in \mathbb{R}^n$$
The reverse is not true in general:

Let us consider, in $\mathbb{R}^2$, the convex subset

$$X = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0 \text{ and } 2x_1 + x_2 \leq 1\}.$$ 

and the two players generalized Nash problem defined by $X$ and

the loss $\theta_1(x_1, x_2) = (x_1 - 2)^2$ and $\theta_2(x_1, x_2) = (x_2 - 2)^2$. 
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The solution set of the GNEP is

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GNEP = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, \ x_2 \geq 0 \quad \text{and} \quad 2x_1 + x_2 = 1 \}.
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The solution set of the GNEP is

$$GNEP = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0 \text{ and } 2x_1 + x_2 = 1\}$$

whereas the variational inequality $S(F, X)$ has the point $(\bar{x}_1, \bar{x}_2) = (0, 1)$ as unique solution.
To simplify the notations, we will denote, for any $\nu$ and any $x \in \mathbb{R}^n$, by $S_\nu(x)$ and $A_\nu(x^{-\nu})$ the subsets of $\mathbb{R}^{n_\nu}$

$$S_\nu(x) = S^a_{\theta_\nu(\cdot, x^{-\nu})}(x^\nu) \quad \text{and} \quad A_\nu(x^{-\nu}) = \arg \min_{\mathbb{R}^{n_\nu}} \theta_\nu(\cdot, x^{-\nu}).$$

In order to construct the variational inequality problem we define the following set-valued map $N^a_{\theta} : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ which is described,

for any $x = (x^1, \ldots, x^p) \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_p}$, by

$$N^a_{\theta}(x) = F_1(x) \times \ldots \times F_p(x),$$

where $F_\nu(x) = \begin{cases} B_\nu(0, 1) & \text{if } x^\nu \in A_\nu(x^{-\nu}) \\ \overline{\text{co}}(N^a_{\theta_\nu}(x^\nu) \cap S_\nu(0, 1)) & \text{otherwise} \end{cases}$

The set-valued map $N^a_{\theta}$ has nonempty convex compact values.
In the following we assume that $X$ is a given nonempty subset $X$ of $\mathbb{R}^n$, such that for any $i$, the set $X_i(x^{-i})$ is given as

$$X_i(x^{-i}) = \{ x^i \in \mathbb{R}^n : (x^i, x^{-i}) \in X \}.$$ 

**Theorem**

*Let us assume that, for any $i$, the function $\theta_i$ is continuous and quasiconvex with respect to the $i$-th variable. Then every solution of $S(N_\theta^a, X)$ is a solution of the GNEP.*
In the following we assume that $X$ is a given nonempty subset $X$ of $\mathbb{R}^n$, such that for any $i$, the set $X_i(x^{-i})$ is given as

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**Theorem**

Let us assume that, for any $i$, the function $\theta_i$ is continuous and quasiconvex with respect to the $i$-th variable. Then every solution of $S(N^a_\theta, X)$ is a solution of the GNEP.

Note that the link between GNEP and variational inequality is valid even if the constraint set $X$ is neither convex nor compact.
Lemma

Let $\nu \in \{1, \ldots, p\}$. If the function $\theta_{\nu}$ is continuous quasiconvex with respect to the $\nu$-th variable, then,

$$0 \in F_\nu(\bar{x}) \iff \bar{x}^\nu \in A_\nu(\bar{x}^{-\nu}).$$

**Proof.** It is sufficient to consider the case of a point $\bar{x}$ such that $\bar{x}^\nu \not\in A_\nu(\bar{x}^{-\nu})$. Since $\theta_{\nu}(\cdot, \bar{x}^{-\nu})$ is continuous at $\bar{x}^\nu$, the interior of $S_\nu(\bar{x})$ is nonempty. Let us denote by $K_\nu$ the convex cone

$$K_\nu = N^\circ_{\theta_{\nu}}(\bar{x}^\nu) = (S_\nu(\bar{x}) - \bar{x}^\nu)^\circ.$$

By quasiconvexity of $\theta_{\nu}$, $K_\nu$ is not reduced to $\{0\}$. Let us first observe that, since $S_\nu(\bar{x})$ has a nonempty interior, $K_\nu$ is a pointed cone, that is $K_\nu \cap (-K_\nu) = \{0\}$.

Now let us suppose that $0 \in F_\nu(\bar{x})$. By Caratheodory theorem, there exist vectors $v_i \in [K_\nu \cap S_\nu(0, 1)]$, $i = 1, \ldots, n + 1$ and scalars $\lambda_i \geq 0$, $i = 1, \ldots, n + 1$ with

$$\sum_{i=1}^{n+1} \lambda_i = 1 \text{ and } 0 = \sum_{i=1}^{n+1} \lambda_i v_i.$$
Since there exists at least one $r \in \{1, \ldots, n + 1\}$ such that $\lambda_r > 0$ we have

$$v_r = - \sum_{i=1, i \neq r}^{n+1} \frac{\lambda_i}{\lambda_r} v_i$$

which clearly shows that $v_r$ is an element of the convex cone $-K_\nu$. But $v_r \in S_\nu(0, 1)$ and thus $v_r \neq 0$. This contradicts the fact that $K_\nu$ is pointed and the proof is complete. $\blacksquare$
Proof. Let us consider $\bar{x}$ to be a solution of $S(N^a_{\theta}, X)$. There exists $\nu \in N^a_{\theta}(\bar{x})$ such that

$$\langle \nu, y - \bar{x} \rangle \geq 0, \quad \forall y \in X. \quad (*)$$

Let $\nu \in \{1, \ldots, p\}$.
If $\bar{x}^\nu \in A^\nu(\bar{x}^-\nu)$ then obviously $\bar{x}^\nu \in Sol^\nu(\bar{x}^-\nu)$.
Otherwise $\nu^\nu \in F^\nu(\bar{x}) = co(N^a_{\theta^\nu}(\bar{x}^-\nu) \cap S^\nu(0, 1))$. Thus, according to Lemma 2, there exist $\lambda > 0$ and $u^\nu \in N^a_{\theta^\nu}(\bar{x}^-\nu) \setminus \{0\}$ satisfying $\nu^\nu = \lambda u^\nu$.

Now for any $x^\nu \in X^\nu(\bar{x}^-\nu)$, consider $y = (\bar{x}^1, \ldots, \bar{x}^\nu-1, x^\nu, \bar{x}^\nu+1, \ldots, \bar{x}^p)$.

From $(*)$ one immediately obtains that $\langle u^\nu, x^\nu - \bar{x}^\nu \rangle \geq 0$. Since $x^\nu$ is an arbitrary element of $X^\nu(\bar{x}^-\nu)$, we have that $\bar{x}^\nu$ is a solution of $S(N^a_{\theta^\nu} \setminus \{0\}, X^\nu(\bar{x}^-\nu))$ and therefore, according to Prop. 4,

$$\bar{x}^\nu \in Sol^\nu(\bar{x}^-\nu)$$

Since $\nu$ was arbitrarily chosen we conclude that $\bar{x}$ solves the GNEP.
Theorem

Let us suppose that, for any $i$, the loss function $\theta_i$ is continuous and semistrictly quasiconvex with respect to the $i$-th variable. Further assume that the set $X$ is a nonempty convex subset of $\mathbb{R}^N$. Then

any solution of the variational inequality $S(N_{\hat{\theta}}, X)$ is a solution of the GNEP

and

any solution of the GNEP is a solution of the quasi-variational inequality $QVI(N_{\hat{\theta}}, X)$

where $X$ stands for the set-valued map defined on $\mathbb{R}^2$ by

$$X(x) = \prod_{i=1}^{p} X_i(x^{-i})$$
Existence of GNEP

Based on the previous developments it is possible to obtain an existence result for general GNEP in an elegant and direct way.

**Theorem**

Assume that for every player \( \nu \), the loss function \( \theta_{\nu} \) is continuous on \( \mathbb{R}^n \) and semistrictly quasiconvex with respect to the \( \nu \)-th variable. If the set \( X \) involved in the Rosen law is nonempty, convex and compact, then the generalized Nash equilibrium problem admits at least a solution.

**Proof:** According to sufficient optimality condition, it is sufficient to show that the variational inequality \( VI(N_{\theta}^a, X) \) admits a solutions. But \( X \) is nonempty convex compact and the operator \( N_{\theta}^a \) has nonempty convex compact values included in the compact set \( \overline{B}_1(0, 1) \times \cdots \times \overline{B}_p(0, 1) \). Therefore, according to classical existence result, the variational inequality \( VI(N_{\theta}^a, X) \) admits at least one solution, provided that the operator \( N_{\theta}^a \) is closed.

Looking to the definition of \( N_{\theta}^a \), we thus need to show that, for any \( \nu \), the set-valued map \( F_{\nu} \) defined in (??) has a closed graph. So let \( \nu \in \{1, \ldots, p\} \) and consider a sequence \( (x_k)_k \) of \( X \) converging to \( \bar{x} \) and, for any \( k \in \mathbb{N} \), \( v_{\nu}^k \in F_{\nu}(x_k) \) be such that the sequence \( (v_{\nu}^k)_k \) converge (to \( \bar{v}_{\nu} \)).
Proof (continued)

If $\bar{x}^\nu$ is an element of $A^\nu(\bar{x}^{-\nu})$ then obviously $\bar{v}^\nu$, as a limit of elements of the closed set $\bar{B}_\nu(0,1)$, is an element of $F_\nu(\bar{x}) = \bar{B}_\nu(0,1)$.

If $\bar{x}^\nu \not\in A^\nu(\bar{x}^{-\nu})$ then, due to the continuity of the loss function $\theta^\nu$ on $\mathbb{R}^n$, one have, for $k$ large enough, $x_k^\nu \not\in A^\nu(x_k^{-\nu})$. This implies that $v_k^\nu \in \text{conv}(N_{\theta^\nu}(x_k^\nu) \cap S_\nu(0,1))$ and therefore, by Caratheodory theorem, there exist $\lambda^i_k \in [0,1], i = 1, \ldots, n_\nu + 1$ and $u^i_k \in N_{\theta^\nu}(x_k^\nu) \cap S_\nu(0,1), i = 1, \ldots, n_\nu + 1$ such that

$$v_k^\nu = \sum_{i=1}^{n_\nu+1} \lambda^i_k u^i_k \quad \text{and} \quad \sum_{i=1}^{n_\nu+1} \lambda^i_k = 1.$$

Considering possibly subsequences, for any $i = 1, \ldots, n_\nu + 1$, the sequences $(\lambda^i_k)_k$ and $(u^i_k)_k$ converge, respectively, to $\bar{\lambda}^i \in [0,1]$ and to $\bar{u}^i \in S_\nu(0,1)$. Moreover

$$\bar{v}^\nu = \sum_{i=1}^{n_\nu+1} \bar{\lambda}^i \bar{u}^i \quad \text{and} \quad \sum_{i=1}^{n_\nu+1} \bar{\lambda}^i = 1.$$

In order to prove that $\bar{v}^\nu$ is an element of $F_\nu(\bar{x}) = \text{conv}(N_{\theta^\nu}(\bar{x}^\nu) \cap S_\nu(0,1))$ it is sufficient to show that, for any $i$, $\bar{u}^i \in N_{\theta^\nu}(\bar{x}^\nu)$. But this is an immediate consequence of the continuity of $\theta^\nu$. Indeed, for any $y^\nu \in \mathbb{R}^{n_\nu}$ such that $\theta^\nu(y^\nu, \bar{x}^{-\nu}) < \theta^\nu(\bar{x}^\nu, \bar{x}^{-\nu})$, one have $\theta^\nu(y^\nu, x_k^{-\nu}) < \theta^\nu(x_k^\nu, x_k^{-\nu})$ for $k$ large enough.

Therefore, by definition of $u^i_k$ and (??), $\langle u^i_k, y^\nu - x_k^\nu \rangle \leq 0$. Taking the limit as $k \to \infty$, $\langle \bar{u}^i, y^\nu - \bar{x}^\nu \rangle \leq 0$ which shows that $\bar{u}^i \in N_{\theta^\nu}(\bar{x}^\nu)$ proving the claim and completing the proof. \square
IV - Reformulation of GNEP using gap functions
Suppose that for any $i$ and any $x^{-i} \in \mathbb{R}^{n-i}$, function $\theta_i(\cdot, x^{-i})$ is continuously differentiable.

Let us consider a special form of the sets $X_i(x^{-i})$. This form was originally used by Rosen in his fundamental paper (1965):

**Given a nonempty convex subset $X$ of $\mathbb{R}^n$, for any $\nu$, the set $X_i(x^{-i})$ is given as**

$$X_i(x^{-i}) = \{x^i \in \mathbb{R}^n_i : (x^i, x^{-i}) \in X\}.$$ 

Define the Nikaido-Isoda function $\Psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\Psi(x, y) = \sum_{i=1}^{N} \left[ \theta_i(x^i, x^{-i}) - \theta_i(y^i, x^{-i}) \right].$$
Theorem

We suppose that the loss functions are continuous on $\mathbb{R}^n$ and that $\theta_i$ are convex with respect to the $i$-th variable. Let $\alpha > 0$ and

$$\Psi_\alpha(x, y) = \sum_{i=1}^{n} \left[ \theta_i(x^i, x^{-i}) - \theta_i(y^i, x^{-i}) + \frac{\alpha}{2} \|x^i - y^i\|^2 \right]$$

i) For all $x \in \mathbb{R}^n$, the application $\Psi_\alpha(x, \cdot)$ admits an unique minimizer over $S(x)$, denoted by $y(x)$.

ii) $\bar{x}$ is a solution of GNEP $\iff y(\bar{x}) = \bar{x}$.

iii) Let $V_\alpha(x) = - \inf_{y \in S(x)} \Psi_\alpha(x, y)$. Then $V_\alpha$ is a gap function for GNEP.

This function $\Psi_\alpha$ is called regularized Nikaido-Isoda-function.

If the functions $\theta_i$ are not convex then the above function $V_\alpha$ is no longer a function gap for GNEP.

Let $N = 2$, $n_1 = n_2 = 1$, $\theta_1(x_1, x_2) = x_1|x_1|$, $\theta_2 = 0$ and $X_1(x_2) = [-1; 1]$, $X_2(x_1) = \mathbb{R}$. Then $V_\alpha(0, 0) = 0$ but $\theta_1(-1, 0) = -1 < \theta_1(0, 0)$, which prove that $(0, 0)$ isn’t a solution of GNEP.
Theorem

We suppose that the loss functions $\theta_i$ are continuous on $\mathbb{R}^N$ and semistrictly quasiconvex with respect to the $i$-th variable. Let $\alpha > 0$ and

$$
\Psi_\alpha(x, y) = \sup_{x^* \in N^a_{\theta}(x)} \langle x^*, y - x \rangle + \frac{\alpha}{2} \| x - y \|^2
$$

i) For all $x \in \mathbb{R}^n$, the application $\Psi_\alpha(x, \cdot)$ admits an unique minimizer over $S(x)$, denoted by $y(x)$.

ii) $\bar{x}$ is a solution of GNEP iif $y(\bar{x}) = \bar{x}$.

iii) Let $V_\alpha(x) = -\inf_{y \in S(x)} \Psi_\alpha(x, y)$. For all $\bar{x} \in X$, $V_\alpha(\bar{x}) \geq 0$, and $\bar{x}$ is a solution of GNEP $\iff V_\alpha(\bar{x}) = 0$.