Dual optimality conditions for MPECs

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Mathematical Program with Equilibrium Constraints (MPEC) 

Mathematical Program with Equilibrium Constraints (MPEC): 

\[
\min \{ \varphi(x, y) | 0 \in F(x, y) + N_{\Gamma}(y) \}
\]

\[\varphi : \mathbb{R}^{n+m} \to \mathbb{R}, \quad F : \mathbb{R}^{n+m} \to \mathbb{R}^m \text{ locally Lipschitz}, \quad \Gamma \subseteq \mathbb{R}^m \text{ closed} \]

\[N \text{ Mordukhovich normal cone} \]

Optimization problem, whose constraints are given by the solution of a generalized equation.

Special instance: **Bilevel Problem** (optimistic form): 

\[
\min \{ \varphi(x, y) | y \in \arg\min \{ f(x, y) | y \in \Gamma \} \}
\]

Here,

\[
F(x, y) = \begin{cases} 
\nabla_y f(x, y) & \text{single-valued, continuously differentiable} \\
\nabla_y f(x, y) & \text{single-valued, locally Lipschitz} \\
\partial_y f(x, y) & \text{set-valued}
\end{cases}
\]

if \( f \in C^2 \)

if \( f \in C^{1,1} \)

if \( f \in C^{0,1} \)
Mathematical Program with Complementarity Constraints (MPCC)

MPEC: \( \min \{ \varphi(x, y) | 0 \in F(x, y) + N_\Gamma(y) \} \quad (\varphi, F \in C^1) \)

Assume that

- \( \Gamma \) is described by inequalities: \( \Gamma = \{ y \in \mathbb{R}^m | g_j(y) \leq 0 \quad j = 1, \ldots, q \} \quad (g \in C^1) \)
- \( \Gamma \) satisfies some basic constraint qualification (e.g., LICQ, MFCQ, Slater etc.)

Then, for \( y \in \Gamma \),

\[
N_\Gamma(y) = \{ v \in \mathbb{R}^m | \exists \lambda \in \mathbb{R}^m_+ : v = [\nabla g(y)]^T \lambda, \quad g(y)^T \lambda = 0 \}
\]

Reformulation of MPEC as a **Mathematical Program with Complementarity Constraints** (MPCC):

MPCC: \( \min_{x, y, \lambda} \{ \varphi(x, y) | F(x, y) + [\nabla g(y)]^T \lambda = 0, \quad g(y)^T \lambda = 0, \quad g(y) \leq 0, \quad \lambda \geq 0 \} \)

Optimization problem in \((x, y, \lambda)\).

\((\bar{x}, \bar{y})\) local solution of MPEC \( \Rightarrow \) \((\bar{x}, \bar{y}, \bar{\lambda})\) local solution of MPCC for all feasible \( \bar{\lambda} \)

\((\bar{x}, \bar{y}, \bar{\lambda})\) local solution of MPCC \( \nRightarrow \) \((\bar{x}, \bar{y})\) local solution of MPEC

(counterexample Dempe/Dutta 2012)
Equilibrium Problem with Equilibrium Constraints (EPEC)

Let \( x = (x_1, \ldots, x_N) \in \mathbb{R}^{Nn} \) denote the decision vectors of \( N \) players in a generalized Nash game.

**Equilibrium Problem with Equilibrium Constraints (EPEC):**

\[
\min_{x_i, y} \left\{ \varphi_i(x, y) \mid 0 \in F(x, y) + N_\Gamma(y) \right\} \quad (i = 1, \ldots, N)
\]

Coupled system of MPECs, which each player tries to solve given the decisions of competitors.

**Definition**

\((\bar{x}, \bar{y})\) is a solution of EPEC, if for each \( i = 1, \ldots, N \), the \((\bar{x}_i, \bar{y})\) are solutions of the MPEC

\[
\min_{x_i, y} \left\{ \varphi_i(\bar{x}_i, x_i, y) \mid 0 \in F((\bar{x}_i, x_i, y), y) + N_\Gamma(y) \right\}
\]

Here, we have used the convention \((\bar{x}_{-i}, x_i) := (\bar{x}_1, \ldots, \bar{x}_{i-1}, x_i, \bar{x}_{i+1}, \ldots, \bar{x}_N)\).
Structure of an ISO-regulated electricity spot market problem

Producer
- maximizes profit

ISO
- minimizes total costs
- subject to demand satisfaction in the network

Producer
- maximizes profit

Demand satisfaction in the network:
\[ p + B t \geq d, \quad p \geq 0, \quad -\bar{t} \leq t \leq \bar{t} \quad (DS) \]
p, t = \text{produced and transmitted power}

Bid and cost functions for producer \( i \):
\[ \alpha_i p_i + \beta_i p_i^2, \quad \gamma_i p_i + \delta_i p_i^2 \]

ISO-problem:
\[ \min_{p,t} \{ \sum_{i=1}^{N} \alpha_i p_i + \beta_i p_i^2 \mid (DS) \} \quad (ISO) \]

Market clearing price for producer \( i \):
\[ \alpha_i + 2\beta_i \bar{p}_i \quad \text{with} \ \bar{p} \ \text{the solution to (ISO)} \]

Profit of producer \( i \):
\[ f_i(\alpha, \beta, p, t) := (\alpha_i - \gamma_i)p_i + (2\beta_i - \delta_i)p_i^2 \]
Formulation as an EPEC

Abstract form of EPEC:

$$\min_{x_i, y} \{ \varphi_i(x, y) | 0 \in F(x, y) + N_{\Gamma}(y) \} \quad (i = 1, \ldots, N)$$

Spot market EPEC:

$$\min_{\alpha_i, \beta_i, p, t} (\alpha_i - \gamma_i)p_i + (2\beta_i - \delta_i)p_i^2$$

subject to

$$(p, t) \in \text{argmin}\left\{ \sum_{i=1}^{N} \alpha_ip_i + \beta_ip_i^2 \mid p + Bt \geq d, \ p \geq 0 \ - \bar{t} \leq t \leq \bar{t} \right\}$$

$$(i = 1, \ldots, N)$$

Put $F(\alpha, \beta, p, t) := \left( \begin{array}{c} \alpha + 2 \text{diag}\beta \ p \\ 0 \end{array} \right)$,
bidding curve (red), true costs (blue), demands (green)
Solution of ISO problem: production (red) and transmission (arrows)
First MPEC iteration: market clearing price and competitors profits
Nonlinear Programming

General nonlinear programming problem with \( f, g, h \in C^1 \):

\[
\min \{ f(x) | x \in M \}
\]

\[ M := \{ x \in \mathbb{R}^n | h_i(x) = 0 \ (i = 1, \ldots, p), \ g_j(x) \leq 0 \ (j = 1, \ldots, q) \} \]

Short hand notation: \( M = \{ x \in \mathbb{R}^n | h(x) = 0, \ g(x) \leq 0 \} \)

Representation of the feasible set as a preimage of a convex cone:

\[
M = G^{-1}(0_p \times \mathbb{R}_q^q) \quad (G := (h, g) : \mathbb{R}^n \to \mathbb{R}^{p+q})
\]

Under certain constraint qualifications, a local solution \( \bar{x} \) of the problem satisfy the KKT-conditions

\[
0 = \nabla f(\bar{x}) + [\nabla h(\bar{x})]^T \mu + [\nabla g(\bar{x})]^T \lambda, \quad \lambda \geq 0, \quad \lambda^T g(\bar{x}) = 0
\]

for Lagrange multipliers \( \mu \in \mathbb{R}^p, \lambda \in \mathbb{R}^q \).
Basic constraint qualifications

Let \( \bar{x} \in M := \{ x \in \mathbb{R}^n | h(x) = 0, g(x) \leq 0 \} \)

\( I(\bar{x}) := \{ i | g_i(\bar{x}) = 0 \} \) (index set of active inequality constraints).

**Definition**

\( M \) satisfies the **Linear Independence Constraint Qualification (LICQ)** at \( \bar{x} \) if the set

\[
\{ \nabla h_j(\bar{x}) \}_{j=1}^{p} \cup \{ \nabla g_i(\bar{x}) \}_{i \in I(\bar{x})}
\]

is linearly independent.

**Definition**

\( M \) satisfies the **Mangasarian-Fromovitz Constraint Qualification (MFCQ)** at \( \bar{x} \) if \( \nabla h(\bar{x}) \) is surjective and there exists some \( \xi \in \mathbb{R}^n \) such that

\[
\nabla h(\bar{x})\xi = 0 \quad \langle \nabla g_i(\bar{x}), \xi \rangle < 0 \quad (i \in I(\bar{x}))
\]
## Theorem (Motzkin’s theorem of the alternative)

For given matrices $A, B$ the following equivalence holds true (with '<' componentwise):

$$
\exists \xi : A\xi = 0, \quad B\xi < 0 \quad \iff \quad \mathcal{F}(\mu, \lambda) : \quad A^T\mu + B^T\lambda = 0, \quad \lambda \geq 0, \quad \lambda \neq 0
$$

## Corollary (Dual form of MFCQ)

MFCQ is satisfied for $\bar{x} \in M := \{x \in \mathbb{R}^n | h(x) = 0, \ g(x) \leq 0\}$ if and only if

$$
[\nabla h(\bar{x})]^T\mu + \sum_{i \in I(\bar{x})} \lambda_i [\nabla g_i(\bar{x})]^T = 0, \quad \lambda \geq 0 \implies \lambda = 0, \mu = 0.
$$
Validity of MFCQ in ISO problem

Let \((p^*, t^*)\) be a solution of the ISO-problem:

\[
\min_{p, t} \left\{ \sum_{i=1}^{N} \alpha_i p_i + \beta_i p_i^2 \mid d - p - Bt \leq 0, -p \leq 0, t - \bar{t} \leq 0, -t - \bar{t} \leq 0 \right\}
\]

One can show, that demand satisfaction at a solution is always active: \(p^* + Bt^* = d\).

We choose partitions \(p^* = (p^*_1, p^*_2)\) and \((t^*_1, t^*_2, t^*_3)\) according to activity: \(p^*_2 = 0\), \(t^*_2 = \bar{t}\), \(t^*_3 = -\bar{t}\).

Then, the following components are active at \((p^*, t^*)\): \(d - p - Bt\), \(-p_2\), \(t_2 - \bar{t}\), \(-t_3 - \bar{t}\).

Jacobian of active components:

\[
J = \begin{pmatrix}
-I & 0 & -B_{1,1} & -B_{1,2} & -B_{1,3} \\
0 & -I & -B_{2,1} & -B_{2,2} & -B_{2,3} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & I & 0 \\
0 & 0 & 0 & 0 & -I \\
\end{pmatrix}
\]

We verify MFCQ in its dual form. Assume that \(J^T \lambda = 0\), for some \(\lambda \geq 0\). Then, columns 1,2 of \(J\) yield \(\lambda_1 = 0\), \(\lambda_2 + \lambda_3 = 0 \Rightarrow (by \ \lambda \geq 0): \lambda_2 = \lambda_3 = 0 \Rightarrow (columns 4,5 of \ J): \lambda_4 = \lambda_5 = 0\).

Consequently, \(\lambda = 0\) and, hence, MFCQ is satisfied.
Possible failure of LICQ in ISO problem

Consider the following network

\[ d_1 \quad t_1 = \hat{t}_1 \quad d_2 \]

If at the second node the producer is inactive, the transmission line is congested and the transmission capacity equals the demand, then LICQ is violated:

\[ p_1 - t_1 = d_1 \]
\[ p_2 + t_1 = d_2 \]
\[ p_2 = 0 \]
\[ t_1 = \hat{t}_1 \]

3 variables \((p_1, p_2, t_1)\) and 4 inequalities active \(\implies\) LICQ violated.

This example is exceptional because the equality \(d_2 = \hat{t}_1\) of problem data can be removed by a slight perturbation. If, however, the lower level constraints are also depending on upper level decisions, stable violations of LICQ can be constructed (Dempe/Dutta 2012).

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Contingent cone and abstract primal stationarity condition

Let $C \subseteq \mathbb{R}^n$ be closed and $\bar{x} \in C$. The **contingent cone** to $C$ at $\bar{x}$ is defined as

$$T_C(\bar{x}) := \{d \in \mathbb{R}^n \mid \exists t_n \downarrow 0 \exists x_n \in C : t_n^{-1}(x_n - \bar{x}) \to d\}$$

**Lemma**

Let $\bar{x}$ be a local solution of the abstract minimization problem $\min \{f(x) \mid x \in C\}$. Then,

$$\langle \nabla f(\bar{x}), d \rangle \geq 0 \ \forall d \in T_C(\bar{x})$$

**Proof.**

Exercise
Let $C \subseteq \mathbb{R}^n$ be closed and $\bar{x} \in C$. The **Fréchet normal cone** to $C$ at $\bar{x}$ is defined as the polar cone to the contingent cone:

$$\hat{N}_C(\bar{x}) := [T_C(\bar{x})]^0 = \{ v^* \in \mathbb{R}^n | \langle v^*, d \rangle \leq 0 \ \forall d \in T_C(\bar{x}) \}$$

Always closed, convex cone. For $C$ convex, $\hat{N}_C$ coincides with normal cone from convex analysis. If $T_C$ strictly contains a halfspace, then $\hat{N}_C = \{0\}$ (Exercise).

Consider the abstract minimization problem

$$\min\{ f(x) | x \in C \}$$

and let $\bar{x}$ be a local solution. Then, as a direct consequence of the primal constraint stationarity condition,

$$0 \in \nabla f(\bar{x}) + \hat{N}_C(\bar{x})$$
A simple exercise

Show that for $x \in \mathbb{R}^p_+$ the following holds true:

$$T_{\mathbb{R}^p_+} (x) = \{ h \in \mathbb{R}^p | h_i \geq 0 \forall i : x_i = 0 \}$$

$$\hat{N}_{\mathbb{R}^p_+} (x) = \{ v^* \in \mathbb{R}^- | v_i^* = 0 \forall i : x_i > 0 \}$$

In particular, we get the complementarity relations

$$v^* \in \hat{N}_{\mathbb{R}^p_+} (x) \iff x \geq 0, v^* \leq 0, x^T v^* = 0$$

Moreover, the following equivalence holds true:

$$v^* \in \hat{N}_{\mathbb{R}^p_+} (x) \iff x \in \hat{N}_{\mathbb{R}^-} (v^*)$$
Guignard Constraint Qualification

**Definition**

Let \( C := G^{-1}(\Theta) \) for \( G \in C^1(\mathbb{R}^n, \mathbb{R}^m) \) and \( \Theta \subseteq \mathbb{R}^m \) closed. \( C \) satisfies the **Guignard Constraint Qualification** (GCQ) at \( \bar{x} \in C \) if

\[
\hat{N}_C(\bar{x}) \subseteq [\nabla G(\bar{x})]^T \hat{N}_\Theta(G(\bar{x}))
\]

**Proposition**

*If in the nonlinear programming problem*

\[
\begin{align*}
\min \{ f(x) | x \in M \}, & \quad M := \{ x \in \mathbb{R}^n | h(x) = 0, \ g(x) \leq 0 \}
\end{align*}
\]

*GCQ is satisfied at local solution \( \bar{x} \in M \), then the KKT conditions hold true.*

**Proof.**

Abstract dual stationarity condition + GCQ:

\[
0 \in \nabla f(\bar{x}) + \hat{N}_M(\bar{x}) \subseteq \nabla f(\bar{x}) + ([\nabla h(\bar{x})]^T | [\nabla g(\bar{x})]^T) \hat{N}_{\{0_p\} \times \mathbb{R}^q} (h(\bar{x}), g(\bar{x}))
\]

\[
= \nabla f(\bar{x}) + ([\nabla h(\bar{x})]^T | [\nabla g(\bar{x})]^T) \left[ \mathbb{R}^p \times \hat{N}_{\mathbb{R}^q} (g(\bar{x})) \right]
\]

\[
\implies \exists \mu \in \mathbb{R}^p, \lambda \in \mathbb{R}^q_+ : 0 = \nabla f(\bar{x}) + [\nabla h(\bar{x})]^T \mu + [\nabla g(\bar{x})]^T \lambda, \quad \lambda \geq 0, \ \lambda^T g(\bar{x}) = 0
\]
Let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued mapping (multifunction).

By $\text{gr} \, \Phi := \{(x, y) \in \mathbb{R}^{m+n} | y \in \Phi(x)\}$ we denote its graph.

Example: Frechet normal cone mapping $\hat{N}_{\mathbb{R}^+} : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. 

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Drawbacks of the Fréchet normal cone

\( \text{gr} \hat{N}_C \) is not closed in general \( \implies \)

Fréchet normal cone at some point may be smaller than limits of Fréchet normal cones from close points.

Dual stationarity condition \( 0 \in \nabla f(\bar{x}) + \hat{N}_C(\bar{x}) \) does not generalize to Lipschitzian \( f \).

GCQ: \( \hat{N}_{G^{-1}}(C)(\bar{x}) \subseteq [\nabla G(\bar{x})]^T \hat{N}_C(G(\bar{x})) \) strongly hinges on surjectivity of \( \nabla G(\bar{x}) \).

**Example**

\[ C := \{(x_1, x_2) | x_2 \geq -|x_1|\}, \quad G(x_1, x_2) := (0, x_2), \quad (\bar{x}_1, \bar{x}_2) := (0, 0) \]

\[ \implies G^{-1}(C) = \{(x_1, x_2) | (0, x_2) \in C\} = \mathbb{R} \times \mathbb{R}_+, \quad \hat{N}_C(G(\bar{x})) = \hat{N}_C(\bar{x}) = \{0\}. \]

GCQ violated due to \( \nabla G(\bar{x}) \) not surjective:

\[ \{0\} \times \mathbb{R}_- = \hat{N}_{G^{-1}}(C)(\bar{x}) \not\subseteq [\nabla G(\bar{x})]^T \hat{N}_C(G(\bar{x})) = \{0\} \]
The Mordukhovich (or: limiting) normal cone

Let \( C \subseteq \mathbb{R}^n \) be closed. We have observed that \( \text{gr} \, \hat{N}_C \) is not necessarily closed.

To overcome this drawback, we may introduce a new normal cone mapping \( N_C \) via the relation

\[
\text{gr} \, N_C = \text{cl} \, \text{gr} \, \hat{N}_C.
\]

**Definition**

Let \( \bar{x} \in C \). Then, the **Mordukhovich normal cone** to \( C \) at \( \bar{x} \) is defined as

\[
N_C(\bar{x}) := \{ x^* | \exists (x_n, x^*_n) \to (\bar{x}, x^*) : x_n \in C, \ x^*_n \in \hat{N}_C(x_n) \}
\]
Let \((\bar{x}, \bar{y}) \in \text{gr } N_{\mathbb{R}^+}\). By our previous exercise and by convexity of \(\mathbb{R}^+_+\), we know that

\[
(\bar{x}, \bar{y}) \in \text{gr } N_{\mathbb{R}^+} (\bar{x}) \iff \bar{y} \in \hat{N}_{\mathbb{R}^+} (\bar{x}) \iff \bar{x} \geq 0, \; \bar{y} \leq 0, \; \bar{x} \cdot \bar{y} = 0
\]

Show that

\[
(\bar{x}, \bar{y}) = (0, 0) \implies N_{\text{gr } N_{\mathbb{R}^+}} (\bar{x}, \bar{y}) = \{(x^*, y^*) | y^* = 0 \text{ if } x^* > 0, \; y^* \geq 0 \text{ if } x^* < 0\}
\]

\[
\bar{x} > 0, \; \bar{y} = 0 \implies N_{\text{gr } N_{\mathbb{R}^+}} (\bar{x}, \bar{y}) = \{(x^*, y^*) | x^* = 0\}
\]

\[
\bar{x} = 0, \; \bar{y} < 0 \implies N_{\text{gr } N_{\mathbb{R}^+}} (\bar{x}, \bar{y}) = \{(x^*, y^*) | y^* = 0\}
\]
Properties of the Mordukhovich normal cone

- always **closed** but in general **not convex**
- contains Fréchet normal cone
- if both coincide, i.e., $N_C(\bar{x}) = \hat{N}_C(\bar{x})$, then the set $C$ is called **regular** at $\bar{x}$.

Example: convex sets

Counter-example: previous slide, Fréchet normal cone (brown color) strictly contained

- commutes with cartesian product: $N_{C_1 \times \ldots \times C_n}(\bar{x}_1, \ldots, \bar{x}_n) = N_{C_1}(\bar{x}_1) \times \ldots \times N_{C_n}(\bar{x}_n)$

**Example (revisited)**

$C := \{(x_1, x_2)|x_2 \geq -|x_1|\} = \text{epi}(-|\cdot|),$ $(\bar{x}_1, \bar{x}_2) := (0, 0)$

\[ N_C(\bar{x}) = \text{gr } (-|\cdot|) . \]
Normal cone and subdifferential

For a function $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ denote by $\text{epi} f := \{(x, t) | t \geq f(x)\}$ its epigraph.

If $f$ is lower semicontinuous (l.s.c), then $\text{epi} f$ is closed.

**Definition**

Let $f : \mathbb{R}^n \rightarrow \overline{\mathbb{R}}$ be l.s.c. Then, its subdifferential at $\bar{x} \in \mathbb{R}^n$ is defined as

$$\partial f(\bar{x}) := \{x^* \in \mathbb{R}^n | (x^*, -1) \in N_{\text{epi} f}\}$$

**Example (revisited)**

Let $C := \{(x_1, x_2) | x_2 \geq -|x_1|\} = \text{epi}(-|\cdot|)$, $(\bar{x}_1, \bar{x}_2) := (0, 0)$

$$\partial (-|\cdot|)(0) = \{-1, 1\} \text{ nonconvex.}$$
Properties of the subdifferential

- coincides with subdifferential of convex analysis in case of convex functions
- coincides with gradient in case of continuously differentiable functions
- smaller than Clarkes subdifferential (equality for regular functions)
- subdifferential of indicator to a set coincides with the normal cone to this set:

\[ \partial i_C(\bar{x}) = N_C(\bar{x}) \quad \text{for } i_C(x) := \begin{cases} 
0 & x \in C \\
\infty & x \notin C 
\end{cases} \]

- compact set for locally Lipschitzian functions

**Theorem (H. 1995)**

*In finite dimensions, the subdifferential of a locally Lipschitzian function can be homeomorphic to any compact set.*

**Proposition**

*Let \( \bar{x} \) be a local minimizer of an l.s.c. function \( f : \mathbb{R}^n \to \mathbb{R} \). Then, \( 0 \in \partial f(\bar{x}) \).*
Theorem (Mordukhovich 1984)

Let $f_1 : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitzian and let $f_2 : \mathbb{R}^n \to \overline{\mathbb{R}}$ be l.s.c. Then,

$$\partial (f_1 + f_2)(\bar{x}) \subseteq \partial f_1(\bar{x}) + \partial f_2(\bar{x}).$$

Corollary (Abstract M-stationarity condition)

Let $f : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitzian and let $\bar{x}$ be a local solution of the optimization problem

$$\min \{ f(x) | x \in C \} \quad (C \text{ closed})$$

Then, $0 \in \partial f(\bar{x}) + N_C(\bar{x}).$

Proof.

$\bar{x}$ is also a local solution of the unconstrained optimization problem $\min f(x) + i_C(x).$

By the optimality condition for unconstrained problems: $0 \in \partial (f + i_C)(\bar{x}).$

$C$ closed $\implies$ $i_C$ l.s.c. $\implies$ sum rule: $0 \in \partial f(\bar{x}) + \partial i_C(\bar{x}) = \partial f(\bar{x}) + N_C(\bar{x}).$

Formula would not apply to Fréchet normal cones.
Coderivative

**Definition**

Let $\Phi : \mathbb{R}^n \Rightarrow \mathbb{R}^m$ have a closed graph. Fix any $(\bar{x}, \bar{y}) \in \text{gr} \Phi$. Then, the coderivative of $\Phi$ at $(\bar{x}, \bar{y})$ is defined as a multifunction $D^*\Phi(\bar{x}, \bar{y}) : \mathbb{R}^m \Rightarrow \mathbb{R}^n$ such that

$$D^*\Phi(\bar{x}, \bar{y})(y^*) := \{x^* | (x^*, -y^*) \in N_{\text{gr} \Phi}(\bar{x}, \bar{y})\}$$

If $\Phi$ is single-valued, then $\bar{y} = \Phi(\bar{x})$ and we simply write $D^*\Phi(\bar{x})$ rather than $D^*\Phi(\bar{x}, \Phi(\bar{x}))$.

If $\Phi \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, then $D^*\Phi(\bar{x}) = [\nabla \Phi(\bar{x})]^T$.

Coderivative generalizes calculus of single-valued differentiable mappings to set-valued, nondifferentiable mappings.

In the example: $D^*\Phi(\bar{x}, \bar{y})(y^*) = \begin{cases} \{0\} & y^* \leq 0 \\ \emptyset & y^* > 0 \end{cases}$

**Proposition (Scalarization formula)**

If $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is locally Lipschitzian, then $D^*\Phi(\bar{x})(y^*) = \partial \langle y^*, \Phi(\bar{x}) \rangle$ for all $y^* \in \mathbb{R}^m$. 

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Exercise

Let $(\bar{x}, \bar{y}) \in \text{gr} \, N_{\mathbb{R}_+}$. Using our previous exercise for computing $N_{\text{gr} \, N_{\mathbb{R}_+}}(\bar{x}, \bar{y})$, show that

If $(\bar{x}, \bar{y}) = (0, 0) \implies D^* N_{\mathbb{R}_+}(\bar{x}, \bar{y})(y^*) = \begin{cases} \mathbb{R} & \text{if } y^* = 0 \\ \{0\} & \text{if } y^* > 0 \\ \mathbb{R}_- & \text{if } y^* < 0 \end{cases}$

If $\bar{x} > 0$, $\bar{y} = 0 \implies D^* N_{\mathbb{R}_+}(\bar{x}, \bar{y})(y^*) = \{0\}$

If $\bar{x} = 0$, $\bar{y} < 0 \implies D^* N_{\mathbb{R}_+}(\bar{x}, \bar{y})(y^*) = \begin{cases} \mathbb{R} & \text{if } y^* = 0 \\ \emptyset & \text{if } y^* \neq 0 \end{cases}$
Preimage formulae for the Mordukhovich normal cone

Let \( \bar{x} \in G^{-1}(C) \) for \( G : \mathbb{R}^n \rightarrow \mathbb{R}^m \) and \( C \subseteq \mathbb{R}^m \).

**Proposition (first preimage formula, see Rockafellar/Wets 1998)**

Let \( C \) be closed, \( G \) be continuously differentiable and \( \nabla G(\bar{x}) \) be surjective. Then,

\[
N_{G^{-1}(C)}(\bar{x}) = [\nabla G(\bar{x})]^T N_C(G(\bar{x}))
\]

**Proposition (second preimage formula, Mordukhovich 1994)**

Let \( C \) be closed, \( G \) be continuous and let the Basic Constraint Qualification

\[
\ker D^* G(\bar{x}) \cap N_C(G(\bar{x})) = \{0\} \quad (BCQ)
\]

be satisfied. Then,

\[
N_{G^{-1}(C)}(\bar{x}) \subseteq D^* G(\bar{x}) [N_C(G(\bar{x}))].
\]

**Proposition (third preimage formula, Mordukhovich 1994)**

Let \( C \) be closed and regular (e.g., \( C \) convex), \( G \) be continuously differentiable and let the Basic Constraint Qualification

\[
\ker [\nabla G(\bar{x})]^T \cap N_C(G(\bar{x})) = \{0\} \quad (BCQ)
\]

be satisfied. Then,

\[
N_{G^{-1}(C)}(\bar{x}) = [\nabla G(\bar{x})]^T N_C(G(\bar{x})).
\]
Example revisited

\(C := \{(x_1, x_2)|x_2 \geq -|x_1|\}, \quad G(x_1, x_2) := (0, x_2), \quad (\bar{x}_1, \bar{x}_2) := (0, 0)\)

\(\implies G^{-1}(C') = \{(x_1, x_2)|(0, x_2) \in C\} = \mathbb{R} \times \mathbb{R}_+\)

\(N_C(G(\bar{x})) = N_C(\bar{x}) = \{(x_1, x_2)|x_2 = -|x_1|\}\).

\(\nabla G(\bar{x}) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\) not surjective, first preimage formula does not apply

\(C\) is not regular due to \(\{0\} = \hat{N}_C(G(\bar{x})) \neq N_C(G(\bar{x})) \implies\) third preimage formula does not apply.

**But:** \(\ker [\nabla G(\bar{x})]^T \cap N_C(G(\bar{x})) = [\mathbb{R} \times \{0\}] \cap \{(x_1, x_2)|x_2 = -|x_1|\} = \{0\}\)

\(\implies\) second preimage formula does apply:

\[N_{G^{-1}(C)}(\bar{x}) \subseteq [\nabla G(\bar{x})]^T N_C(G(\bar{x}))\]

A direct check shows that actually equality holds true here:

\[N_{G^{-1}(C)}(\bar{x}) = \{0\} \times \mathbb{R}_-\]

\([\nabla G(\bar{x})]^T N_C(G(\bar{x})) = \bigcup_{(x_1, x_2)} \{(0, x_2)|x_2 = -|x_1|\} = \{0\} \times \mathbb{R}_-\]
Aim: Compute $D^* N_{\mathbb{R}_+^p}$. Denote by $L$ the permutation mapping defined as

$$L(x_1, \ldots, x_p, y_1, \ldots, y_p) := (x_1, y_1, \ldots, x_p, y_p).$$

Observe that $L$ is regular and $L^{-1} = L = L^T$. Now, with $\Lambda := \text{gr } N_{\mathbb{R}_+^1} \times \cdots \times \text{gr } N_{\mathbb{R}_+^1}$ we get that

$$\text{gr } N_{\mathbb{R}_+^p} = \{(x, y) | y \in N_{\mathbb{R}_+^p}(x)\} \quad \text{(by commutation with cartesian product)}$$

$$= \{(x, y) | (x_i, y_i) \in \text{gr } N_{\mathbb{R}_+^1}, \ i = 1, \ldots, p\} = L^{-1}(\Lambda)$$

The first preimage formula yields:

$$N_{\text{gr } N_{\mathbb{R}_+^p}}(x, y) = N_{L^{-1}(\Lambda)}(x, y) = LN_\Lambda(L(x, y))$$

Hence, $(x^*, y^*) \in N_{\text{gr } N_{\mathbb{R}_+^p}}(x, y) \iff L^{-1}(x^*, y^*) \in N_\Lambda(L(x, y))$

$$\iff (x_1^*, y_1^*, \ldots, x_p^*, y_p^*) \in N_{\text{gr } N_{\mathbb{R}_+}^1}(x_1, y_1) \times \cdots \times N_{\text{gr } N_{\mathbb{R}_+}^1}(x_p, y_p)$$

$$\iff (x_i^*, y_i^*) \in N_{\text{gr } N_{\mathbb{R}_+^1}}(x_i, y_i), \ i = 1, \ldots, p$$
So far, we have that

\[ D^*N_{\mathbb{R}^p_+}(x, y)(y^*) = \{ x^* | (x^*, -y^*) \in N_{gr\mathbb{R}^p_+}(x, y) \} \]

\[ = \{ x^* | (x_i^*, -y_i^*) \in N_{gr\mathbb{R}_+}(x_i, y_i), \ i = 1, \ldots, p \} \]

\[ = \{ x^* | x_i^* \in D^*N_{\mathbb{R}^p_+}(x_i, y_i)(y_i^*) \} \]

Can be explicitly calculated via previous exercise for computation of \( D^*N_{\mathbb{R}_+} \):

\[ D^*N_{\mathbb{R}^p_+}(x, y)(y^*) = \emptyset \text{ if there exists some } i \text{ such that } \bar{x}_i = 0, \bar{y}_i < 0, y_i^* \neq 0 \]

Otherwise:

\[ D^*N_{\mathbb{R}^p_+}(x, y)(y^*) = \begin{cases} x_i^* = 0 & \text{if } x_i = y_i = 0, y_i^* > 0 \\ x^* & \text{or } x_i > 0, y_i = 0 \\ x_i^* \leq 0 & \text{if } x_i = y_i = 0, y_i^* < 0 \end{cases} \]
Lipschitz properties of single-valued mappings

Let $X, Y$ be metric spaces.

$f : X \to Y$ is locally Lipschitz at $\bar{x}$ if there are $L \geq 0, \varepsilon > 0$ such that

$$d(f(x_1), f(x_2)) \leq Ld(x_1, x_2) \quad \forall x_1, x_2 \in B_{\varepsilon}(\bar{x})$$

$f : X \to Y$ is calm at $\bar{x}$ if there are $L \geq 0, \varepsilon > 0$ such that

$$d(f(x), f(\bar{x})) \leq Ld(x, \bar{x}) \quad \forall x \in B_{\varepsilon}(\bar{x})$$

$f$ not locally Lipschitz but calm

![Diagram of a function with a dashed line indicating non-Lipschitz behavior and a solid line indicating calm behavior.]
Hausdorff Lipschitz continuity of set-valued mappings

Let $X, Y$ be metric spaces and $F : X \rightrightarrows Y$ a multifunction.

- Distance between images = distance between sets

Point-to-set distance: $d(y, A) := \inf_{a \in A} d(y, a)$.

Hausdorff distance between closed sets $A, B \subseteq Y$:

$$d_H(A, B) = \max\{\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)\}$$

$F$ is Hausdorff Lipschitz at $\bar{x}$ if there are $L \geq 0, \varepsilon > 0$ such that

$$d_H(F(x_1), F(x_2)) \leq Ld(x_1, x_2) \quad \forall x_1, x_2 \in B_{\varepsilon}(\bar{x})$$

$$d_H(A, A_n) = \infty$$

Hausdorff distance not suited for unbounded sets

Local behavior not well reflected
Aubin property

**Definition**

Let $X, Y$ be metric spaces and $F : X \rightrightarrows Y$ a multifunction. Fix a point $(\bar{x}, \bar{y}) \in \text{gr}F$.

Then, $F$ has the **Aubin property** at $(\bar{x}, \bar{y})$, if there are $L \geq 0, \varepsilon > 0$ such that

$$d(y, F(x_2)) \leq Ld(x_1, x_2) \quad \forall y \in F(x_1) \cap B_\varepsilon(\bar{y}) \quad \forall x_1, x_2 \in B_\varepsilon(\bar{x})$$

From top to bottom:
- infinite slopes for $(x_1, x_2)$ close to $\bar{x}$.
- $(\bar{x}, \bar{y}) \in \text{int gr} F$.
- at most linear growth
- locally empty images for $x$ close to $\bar{x}$
- sublinear growth
Calmness

Definition

Let \( X, Y \) be metric spaces and \( F : X \rightrightarrows Y \) a multifunction. Fix a point \((\bar{x}, \bar{y})\) \(\in\) \(\text{gr} F\).

Then, \( F \) is **calm** at \((\bar{x}, \bar{y})\), if there are \(L \geq 0, \varepsilon > 0\) such that

\[
d(y, F(x)) \leq Ld(x, \bar{x}) \quad \forall y \in F(x) \cap B_{\varepsilon}(\bar{y}) \forall x \in B_{\varepsilon}(\bar{x})
\]

From top to bottom:
- finite slopes for \( x \) close to \( \bar{x} \).
- \( (\bar{x}, \bar{y}) \in \text{int} \text{ gr} F \).
- at most linear growth
- locally images for \( x \) are contained in that for \( \bar{x} \)
- sublinear growth
Comparison of Aubin property and calmness

- both are Lipschitz properties of multifunctions
- Aubin property implies calmness
- Aubin property measures Lipschitz property on arbitrary pairs of arguments close to a fixed one
- calmness measures Lipschitz property only for one arbitrary argument with respect to a fixed argument
- by symmetry, Aubin property is an upper and lower semicontinuity property at the same time (no explosion or implosion of images)
- calmness is just an upper semicontinuity property (no explosion, implosion possible)
- Aubin property fails if images for arguments close to the fixed one are empty
- calmness holds true if images for arguments close to the fixed one are smaller than image at the fixed argument
- Aubin property of a multifunction is the same as **metric regularity** of its inverse
- For single-valued mappings, Aubin property reduces to classical local Lipschitz property
- calmness of canonically perturbed inclusions is equivalent to existence of a **local error bound**:

\[
F(y) := \{ x | g(x) + y \in \Theta \} \quad (g \text{ continuous}, \theta \text{ closed}) \text{ is calm at } (0, \bar{x}) \in \text{gr} F
\]

\[
\iff \exists L, \varepsilon > 0 : d(x, g^{-1}(\Theta)) \leq L d(g(x), \Theta) \forall x \in B_{\varepsilon}(\bar{x})
\]
Theorem (Mordukhovich 1992)

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ have a closed graph. Then, $M$ has the Aubin property at $(\bar{x}, \bar{y}) \in \text{gr} F$ if and only if

$$D^* F(\bar{x}, \bar{y})(0) = \{0\}$$

Since always $(0, 0) \in N_{\text{gr} F}(\bar{x}, \bar{y})$ and, hence, $0 \in D^* F(\bar{x}, \bar{y})(0)$ one has the following

Corollary

$M$ has the Aubin property at $(\bar{x}, \bar{y}) \in \text{gr} F$ if and only if $D^* F(\bar{x}, \bar{y})(0) \subseteq \{0\}$. 
Criterion for Aubin property of smooth constraint systems:

Proposition

\[ F(p) := \{ x \in \mathbb{R}^n | G(x) - p \in \Theta \}, \quad G \in C^1(\mathbb{R}^n, \mathbb{R}^m), \quad G(\bar{x}) \in \Theta \subseteq \mathbb{R}^m \text{ (closed)} \]. Then,

\[ (BCQ) : \quad \text{Ker}[\nabla G(\bar{x})]^T \cap N_\Theta(G(\bar{x})) = \{0\} \iff F \text{ has the Aubin property at } (0, \bar{x}) \]

Proof.

\[ \tilde{G}(p, x) := G(x) - p \implies \text{gr} F = \tilde{G}^{-1}(\Theta) \]

\[ \nabla \tilde{G}(0, \bar{x}) = (-I, \nabla G(\bar{x})), \quad \text{surjective} \implies \text{first preimage formula:} \]

\[ N_{\tilde{G}^{-1}(\Theta)}(0, \bar{x}) = [\nabla \tilde{G}(0, \bar{x})]^T N_\Theta(\tilde{G}(0, \bar{x})) = \begin{pmatrix} -I \\ [\nabla G(\bar{x})]^T \end{pmatrix} N_\Theta(\tilde{G}(0, \bar{x})) \]

\[ (p^*, 0) \in N_{\tilde{G}^{-1}(\Theta)}(0, \bar{x}) \iff \exists z^* \in N_\Theta(G(\bar{x})) : p^* = -z^*, \quad [\nabla G(\bar{x})]^T z^* = 0 \]

\[ \iff -p^* \in \text{Ker}[\nabla G(\bar{x})]^T \cap N_\Theta(G(\bar{x})) \]

\[ D^* F(0, \bar{x})(0) = \{ p^* | (p^*, 0) \in N_{\text{gr} F}(0, \bar{x}) \} = - \left\{ \text{Ker}[\nabla G(\bar{x})]^T \cap N_\Theta(G(\bar{x})) \right\} = \{0\} \]

Apply Mordukhovich criterion.
Aubin property for equality/inequality constraints

Consider again the constraint set of nonlinear programming:

\[ M = \{ x \in \mathbb{R}^n | h(x) = 0, \ g(x) \leq 0 \} \quad (g, h \in C^1) \]

Define the multifunction \( F(p_1, p_2) := \{ x \in \mathbb{R}^n | h(x) = p_1, \ g(x) \leq p_2 \} \)

**Proposition**

Let \( \bar{x} \in M \). Then, MFCQ is satisfied at \( \bar{x} \) if and only if \( F \) has the Aubin property at \( (0, 0, \bar{x}) \).

**Proof.**

\[ F(p) = \{ x \in \mathbb{R}^n | (h(x) - p_1, g(x) - p_2) \in \{0\} \times \mathbb{R}^q \} \]

\( F \) has the Aubin property at \( (0, 0, \bar{x}) \) if and only if (see previous proposition)

\[ \text{Ker}([\nabla h(\bar{x})]^T | [\nabla g(\bar{x})]^T) \cap \left[ N_{\{0\}}(h(\bar{x})) \times N_{\mathbb{R}^q_+}(g(\bar{x})) \right] = \{0\} \]

if and only if

\[ [\nabla h(\bar{x})]^T \mu + [\nabla g(\bar{x})]^T \lambda = 0, \ (\mu, \lambda) \in N_{\{0\}}(h(\bar{x})) \times N_{\mathbb{R}^q_+}(g(\bar{x})) \Rightarrow (\mu, \lambda) = (0, 0) \]

if and only if

\[ [\nabla h(\bar{x})]^T \mu + [\nabla g(\bar{x})]^T \lambda = 0, \ \lambda \geq 0, \ \lambda_i = 0 \ \forall i : g_i(\bar{x}) < 0 \Rightarrow (\mu, \lambda) = (0, 0) \]

Dual form of MFCQ
Failure of MFCQ for complementarity constraints

Consider the constraint set

$$ M = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^q | g(x) \leq 0, \lambda \geq 0, \lambda^T g(x) = 0\} \quad (g \in C^1) $$

**Proposition**

*MFCQ is violated at any $\bar{x} \in M$.*

**Proof.**

Define the multifunction

$$ F(p_1, p_2, p_3) := \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^q | \lambda^T g(x) = p_1, g(x) \leq p_2, -\lambda \leq p_3\} $$

$$ F(n^{-1}, 0, 0) = \{(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^q | \lambda^T g(x) = n^{-1}, g(x) \leq 0, \lambda \geq 0\} = \emptyset \quad \forall n \in \mathbb{N} $$

Images of $F$ empty for $(p_1, p_2, p_3)$ close to $(0, 0, 0) \implies F$ without Aubin property at $(0, 0, 0, \bar{x})$.

By previous proposition, MFCQ is violated at any $\bar{x} \in M$. 

**Corollary**

*The constraint set defined by KKT conditions of a nonlinear program violates MFCQ at all feasible points.*

**Proof.**

Complementarity constraints are subsystem of KKT condition.
Failure of calmness for complementarity constraints

Consider the constraint set

\[ M = \{(x, \lambda) \in \mathbb{R} \times \mathbb{R} | x \leq 0, \ \lambda \geq 0, \ \lambda x = 0\} \]

Then the associated canonical perturbation mapping

\[ F(p_1, p_2, p_3) := \{(x, \lambda) \in \mathbb{R} \times \mathbb{R} | \lambda x = p_1, \ x \leq p_2, \ -\lambda \leq p_3\} \]

fails to be calm at \((0, 0)\):

Observe that \( F(0, 0, 0) = (\mathbb{R}_+ \times \{0\}) \cup (\{0\} \times \mathbb{R}_-) \) and put

\[ \lambda_n := n^{-1/2}, \ x_n := -n^{-1/2} \]

Then, \( d((\lambda_n, x_n), F(0, 0, 0)) = n^{-1/2} \) and

\[ (\lambda_n, x_n) \in F(-n^{-1}, 0, 0). \]

If \( F \) was calm at \((0, 0)\), one should have, however, that

\[ d((\lambda_n, x_n), F(0, 0, 0)) \leq Ln^{-1} \] for some \( L \geq 0. \)
Aubin property for complementarity constraints in variational form I

Complementarity constraints can be equivalently stated in variational form:

\[ g(x) \leq 0, \quad \lambda \geq 0, \quad \lambda^T g(x) = 0 \iff g(x) \in N_{\mathbb{R}_+^p}(\lambda) \iff (\lambda, g(x)) \in \text{gr} \ N_{\mathbb{R}_+^p}(H(\lambda, x)) \]

Hence, the set of complementarity constraints can be written as \( M = H^{-1}(\text{gr} \ N_{\mathbb{R}_+^p}) \).

Canonical perturbation mapping:

\[ \tilde{F}(p) := \{ (\lambda, x) | H(\lambda, x) - p \in \text{gr} \ N_{\mathbb{R}_+^p} \} \]

\[ = \{ (\lambda, x) | g(x) \leq p_2, \quad \lambda \geq p_1, \quad (\lambda - p_1)^T (g(x) - p_2) = 0 \} \]

This is no longer the canonical perturbation mapping \( F \) associated with the equality/inequality description of complementarity constraints.

There is a chance that \( \tilde{F} \) has the Aubin property.
Perturbation mapping \( \tilde{F}(p) = \{(\lambda, x)|H(\lambda, x) - p \in \text{gr } N_{\mathbb{R}^+_p}\} \).

Criterion for Aubin property of smooth constraint systems:

\( \tilde{F} \) has the Aubin property at \((0, \bar{\lambda}, \bar{x}) \iff \text{Ker}[\nabla H(\bar{\lambda}, \bar{x})]^T \cap \text{N}_{\text{gr } N_{\mathbb{R}^+_p}(H(\bar{\lambda}, \bar{x}))} = \{0\} \)

Without loss of generality, we assume that \( g(\bar{x}) = 0 \) (nonactive constraints do not matter locally).

\[ \nabla H(\bar{\lambda}, \bar{x}) = \begin{pmatrix} I & 0 \\ 0 & \nabla g(\bar{x}) \end{pmatrix} \iff \text{Ker}[\nabla H(\bar{\lambda}, \bar{x})]^T = \{(u^*, v^*)|u^* = 0, [\nabla g(\bar{x})]^Tv^* = 0\} \]

By our earlier computations: \( 0 \in D^* N_{\mathbb{R}^+_p}(H(\bar{\lambda}, \bar{x}))(v^*) = D^* N_{\mathbb{R}^+_p}(\bar{\lambda}, 0)(v^*) \) for all \( v^* \).

It follows that \( \text{Ker}[\nabla H(\bar{\lambda}, \bar{x})]^T \cap \text{N}_{\text{gr } N_{\mathbb{R}^+_p}(H(\bar{\lambda}, \bar{x}))} = \{0\} \) if and only if

for all \( v^* \) the relation \([\nabla g(\bar{x})]^Tv^* = 0\) implies that \( v^* = 0 \).

In other words:

\( \tilde{F} \) has the Aubin property at \((0, \bar{\lambda}, \bar{x}) \) if and only if the inequality system \( g(x) \leq 0 \) satisfies LICQ.
Calmness for complementarity constraints in variational form

Calmness, as a weaker property should even has a better chance to hold true.

Perturbation mapping:

\[ \tilde{F}(p) = \{(\lambda, x)|H(\lambda, x) - p \in \text{gr } N_{\mathbb{R}^p_+}\} = \{(\lambda, x)|(\lambda, g(x)) - (p_1, p_2) \in \text{gr } N_{\mathbb{R}^p_+}\} \]

**Theorem (Adam, H., Outrata, 2013)**

Fix any \((\bar{\lambda}, \bar{x})\) such that \(g(\bar{x}) \in N_{\mathbb{R}^p_+}(\bar{\lambda})\). Let \(I := \{i \in \{1, \ldots, p\}|\bar{\lambda}_i > 0\}\).

Then, \(\tilde{F}\) is calm at \((0, \lambda, \bar{x})\) if and only if the multifunction

\[ F^*(p) := \{x \in \mathbb{R}^n|g_i(x) = p_i (i \in I), g_i(x) \leq p_i (i \notin I)\} \]

is calm at \((0, \bar{x})\).

Theorem allows to remove multiplier from the calmness issue for complementarity constraints.

Equivalent characterization just by means of a constraint qualification for the inequality system \(g(x) \leq 0\).

This CQ is much weaker than LICQ which ensures the stronger Aubin property.
Preimage formula under calmness

Theorem (fourth preimage formula, H./Jourani/Outrata 2002)

Let $G : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be locally Lipschitz and let $\Theta \subseteq \mathbb{R}^m$ be closed. If, for some $\bar{x} \in G^{-1}(\Theta)$ the multifunction $\Psi(y) := \{x | G(x) + y \in \Theta\}$ is calm at $(0, \bar{x})$, then the preimage formula

$$N_{G^{-1}(\Theta)}(\bar{x}) \subseteq D^*G(\bar{x}) \left[ N_{\Theta}(G(\bar{x})) \right]$$

holds true.

Corollary (calmness implies GCQ for smooth constraint mappings)

Let $C := G^{-1}(\Theta)$ for $G \in C^1(\mathbb{R}^n, \mathbb{R}^m)$ and $\Theta \subseteq \mathbb{R}^m$ closed and regular. If, for some $\bar{x} \in C$, $\Psi$ as above is calm at $(0, \bar{x})$, then GCQ holds true at $\bar{x}$.

Proof.

$$\hat{N}_C(\bar{x}) = \hat{N}_{G^{-1}(\Theta)}(\bar{x}) \subseteq N_{G^{-1}(\Theta)}(\bar{x}) \subseteq D^*G(\bar{x}) \left[ N_{\Theta}(G(\bar{x})) \right]$$

$$= \left[ \nabla G(\bar{x}) \right]^T N_{\Theta}(G(\bar{x})) = \left[ \nabla G(\bar{x}) \right]^T \hat{N}_{\Theta}(G(\bar{x}))$$

Corollary (KKT conditions)

In the nonlinear program one has that $\Theta = \{0\} \times \mathbb{R}^q_+$ (regular), hence

$LICQ \implies MFCQ \iff Aubin property \implies calmness \implies GCQ \implies KKT$ conditions
Abstract M-stationarity conditions for general MPECs I

Consider the general MPEC

\[ \min \{ \varphi(x, y) | 0 \in F(x, y) + N_{\Gamma}(y) \} \quad (*) \]

with \( \varphi : \mathbb{R}^{n+m} \to \mathbb{R} \), \( F : \mathbb{R}^{n+m} \to \mathbb{R}^m \) locally Lipschitz and \( \Gamma \subseteq \mathbb{R}^m \) closed.

With \( H(x, y) := (y, -F(x, y)) \) one has the equivalence

\[ 0 \in F(x, y) + N_{\Gamma}(y) \iff H(x, y) \in \text{gr} N_{\Gamma} \iff (x, y) \in H^{-1}(\text{gr} N_{\Gamma}) \]

MPEC (*) may be rewritten as

\[ \min \{ \varphi(x, y) | (x, y) \in H^{-1}(\text{gr} N_{\Gamma}) \} \quad (**) \]

If \((\bar{x}, \bar{y})\) is local solution to (*) (or (**)), then (abstract M-stationarity conditions):

\[ 0 \in \partial \varphi(\bar{x}, \bar{y}) + N_{H^{-1}(\text{gr} N_{\Gamma})}(\bar{x}, \bar{y}) \]

If \( \Psi(p) := \{ (x, y) | H(x, y) + p \in \text{gr} N_{\Gamma} \} \) is calm at \((0, \bar{x}, \bar{y})\), then (fourth preimage formula)

\[ N_{H^{-1}(\text{gr} N_{\Gamma})}(\bar{x}, \bar{y}) \subseteq D^* H(\bar{x}, \bar{y}) [N_{\text{gr} N_{\Gamma}}] \]
Abstract M-stationarity conditions for general MPECs II

We obtain

**Lemma**

If \((\bar{x}, \bar{y})\) is local solution to

\[
\min \{\varphi(x, y) | 0 \in F(x, y) + N_\Gamma(y)\}
\]

and the mapping \(\Psi(p) := \{(x, y) | H(x, y) + p \in \text{gr} N_\Gamma\}\) is calm at \((0, \bar{x}, \bar{y})\), then

\[
0 \in \partial \varphi(\bar{x}, \bar{y}) + D^* H(\bar{x}, \bar{y}) \left[ N_{\text{gr} N_\Gamma} (H(\bar{x}, \bar{y})) \right]
\]

Recalling that \(H(x, y) = (y, -F(x, y))\), one may rewrite \(\Psi\) as

\[
\Psi(p_1, p_2) := \{(x, y) | p_2 \in F(x, y) + N_\Gamma(y + p_1)\}
\]

**Lemma (Outrata 2009)**

\(\Psi\) calm at \((0, \bar{x}, \bar{y})\) \(\iff\) \(\tilde{\Psi}(p) := \{(x, y) | p \in F(x, y) + N_\Gamma(y)\}\) calm at \((0, \bar{x}, \bar{y})\).

\(\tilde{\Psi}\) corresponds to canonical perturbation of the generalized equation.
Abstract M-stationarity conditions for general MPECs III

For $H(x, y) = (y, -F(x, y))$ put $H := (H_1, H_2)$. Scalarization formula and sum rule yield

$$D^* H(\bar{x}, \bar{y})(u^*, v^*) = \partial \langle (u^*, v^*), (H_1, H_2) \rangle(\bar{x}, \bar{y})$$
$$\subseteq \partial \langle u^*, H_1 \rangle(\bar{x}, \bar{y}) + \partial \langle v^*, H_2 \rangle(\bar{x}, \bar{y})$$
$$= \{(0, u^*)\} + D^* H_2(\bar{x}, \bar{y})(v^*)$$
$$= \{(0, u^*)\} + D^* (-F)(\bar{x}, \bar{y})(v^*)$$

Theorem

If $(\bar{x}, \bar{y})$ is a local solution to the MPEC with Lipschitz data

$$\min\{\varphi(x, y)|0 \in F(x, y) + N_\Gamma(y)\}$$

and the mapping $\tilde{\Psi}(p) := \{(x, y)|p \in F(x, y) + N_\Gamma(y)\}$ is calm at $(0, \bar{x}, \bar{y})$, then there exists $(u^*, v^*) \in N_{grN_\Gamma}(\bar{y}, -F(\bar{x}, \bar{y}))$ such that

$$0 \in \partial \varphi(\bar{x}, \bar{y}) + \{(0, u^*)\} + D^* (-F)(\bar{x}, \bar{y})(v^*)$$
Consider the smooth MPEC

\[
\min \{ \varphi(x, y) | 0 \in F(x, y) + N_{\Gamma}(y) \} \quad (*)
\]

with \( \varphi : \mathbb{R}^{n+m} \to \mathbb{R} \), \( F : \mathbb{R}^{n+m} \to \mathbb{R}^m \) continuously differentiable and \( \Gamma \subseteq \mathbb{R}^m \) closed. Then,

\[
\partial \varphi(\bar{x}, \bar{y}) = (\nabla_x \varphi(\bar{x}, \bar{y}), \nabla_y \varphi(\bar{x}, \bar{y}))
\]

\[
D^*(-F)(\bar{x}, \bar{y})(v^*) = (-[\nabla_x F(\bar{x}, \bar{y})]^T v^*, -[\nabla_y F(\bar{x}, \bar{y})]^T v^*)
\]

M-stationarity conditions for MPEC with Lipschitz data: \( \exists (u^*, v^*) \in N_{\text{gr}N_{\Gamma}}(\bar{y}, -F(\bar{x}, \bar{y})) : \)

\[
0 = \nabla_x \varphi(\bar{x}, \bar{y}) - [\nabla_x F(\bar{x}, \bar{y})]^T v^*
\]

\[
0 = \nabla_y \varphi(\bar{x}, \bar{y}) - [\nabla_y F(\bar{x}, \bar{y})]^T v^* + u^*
\]

By definition, \( (u^*, v^*) \in N_{\text{gr}N_{\Gamma}}(\bar{y}, -F(\bar{x}, \bar{y})) \iff u^* \in D^* N_{\Gamma}(\bar{y}, -F(\bar{x}, \bar{y}))(-v^*) \)

\( \implies \) compact form: \( \exists v^* \in \mathbb{R}^m : \)

\[
0 = \nabla_x \varphi(\bar{x}, \bar{y}) - [\nabla_x F(\bar{x}, \bar{y})]^T v^*
\]

\[
0 \in \nabla_y \varphi(\bar{x}, \bar{y}) - [\nabla_y F(\bar{x}, \bar{y})]^T v^* + D^* N_{\Gamma}(\bar{y}, -F(\bar{x}, \bar{y}))(-v^*)
\]
Summarizing, we have:

**Theorem (Ye/Ye 1997)**

If \((\bar{x}, \bar{y})\) is a local solution to the MPEC with smooth data

\[
\min \{ \varphi(x, y) | 0 \in F(x, y) + N_{\Gamma}(y) \}
\]

and the mapping \(\tilde{\Psi}(p) := \{(x, y) | p \in F(x, y) + N_{\Gamma}(y)\}\) is calm at \((0, \bar{x}, \bar{y})\),

then there exists an MPEC multiplier \(v^* \in \mathbb{R}^m\) such that

\[
\begin{align*}
0 &= \nabla_x \varphi(\bar{x}, \bar{y}) + [\nabla_x F(\bar{x}, \bar{y})]^T v^* \\
0 &\in \nabla_y \varphi(\bar{x}, \bar{y}) + [\nabla_y F(\bar{x}, \bar{y})]^T v^* + D^* N_{\Gamma}(\bar{y}, -F(\bar{x}, \bar{y}))(v^*)
\end{align*}
\]

Benefits of variational-analytic approach:

- applies to general Lipschitz data
- \(\Gamma\) not necessarily described by inequality constraints
- weak assumption (calmness)

Challenges for application of formula: check calmness of \(\tilde{\Psi}\) and compute coderivative \(D^* N_{\Gamma}\).
Verification of calmness for $\tilde{\Psi}(p) := \{(x, y) | p \in F(x, y) + N_{\Gamma}(y)\}$

We will discuss three options:

- calmness via Aubin property (either direct or by Mordukhovich criterion)
- calmness via polyhedrality
- calmness via Aubin property and calmness for subsystems
Verification via Mordukhovich criterion for Aubin property

The canonical perturbation mapping \( \tilde{\Psi}(p) := \{(x, y) | p \in F(x, y) + N_{\Gamma}(y)\} \) will be calm if the graphical perturbation mapping \( \hat{\Psi}(p_1, p_2) := \{(x, y) | (y, -F(x, y) - (p_1, p_2) \in \text{gr } N_{\Gamma}\} \) is calm (observe that \( \hat{\Psi}(0, p_2) = \tilde{\Psi}(p_2) \)). In particular, the Aubin property of \( \hat{\Psi} \) implies the calmness of \( \tilde{\Psi} \).

**Proposition**

Let \( F \) be continuously differentiable and let \( (\bar{x}, \bar{y}) \) be such that \( 0 \in F(\bar{x}, \bar{y}) + N_{\Gamma}(\bar{y}) \). Then, \( \hat{\Psi} \) has the Aubin property at \( (0, \bar{x}, \bar{y}) \) if and only if the following implication holds true:

\[
[\nabla_x F(\bar{x}, \bar{y})]^T v^* = 0, \quad [\nabla_y F(\bar{x}, \bar{y})]^T v^* \in D^* N_{\Gamma}(\bar{y}, -F(\bar{x}, \bar{y}))(v^*) \implies v^* = 0
\]

**Proof.**

By the criterion for the Aubin property of smooth constraint systems, \( \hat{\Psi} \) has the Aubin property at \( (0, \bar{x}, \bar{y}) \) if and only if

\[
\text{Ker} \begin{pmatrix} 0 & -[\nabla_x F(\bar{x}, \bar{y})]^T \\ I & -[\nabla_y F(\bar{x}, \bar{y})]^T \end{pmatrix} \cap N_{\text{gr } N_{\Gamma}}(\bar{y}, -F(\bar{x}, \bar{y})) = \{0\}.
\]

which is equivalent to the implication

\[
[\nabla_x F(\bar{x}, \bar{y})]^T v^* = 0, \quad u^* - [\nabla_y F(\bar{x}, \bar{y})]^T v^* = 0, \quad u^* \in D^* N_{\Gamma}(\bar{y}, -F(\bar{x}, \bar{y}))(v^*)
\implies u^* = 0, \quad v^* = 0. \quad \text{This yields the assertion.}
\]

Result would follow from \( \nabla_x F(\bar{x}, \bar{y}) \) surjective. A weaker condition will require computation of \( D^* N_{\Gamma} \).
Direct verification of Aubin property

As before, consider the canonical perturbation mapping \( \tilde{\Psi}(p) := \{(x, y) | p \in F(x, y) + N_\Gamma(y)\} \).

**Example**

Let \( F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \) be given by \( F(x, y_1, y_2) := (0, 1) \) and \( \Gamma := \{y \in \mathbb{R}^2 | y_2 \geq 0, y_2 \geq y_1^2\} \).

We verify the Aubin property of \( \tilde{\Psi} \) at the point \((\bar{p}_1, \bar{p}_2, \bar{x}, \bar{y}_1, \bar{y}_2) := (0, 0, 0, 0, 0)\).

\[ \tilde{\Psi}(p_1, p_2) := \{(x, y_1, y_2) | (p_1, p_2 - 1) \in N_\Gamma(y)\} \]

If \( y \in \text{int} \, \Gamma \), then \( N_\Gamma(y) = \{0\} \) which cannot happen for \((p_1, p_2)\) close to \((\bar{p}_1, \bar{p}_2)\).

Therefore, \( y \in \text{bd} \, \Gamma \) for all \((x, y) \in \tilde{\Psi}(p)\) and \( p \) close to \( \bar{p} \). In particular, \( y_2 = y_1^2 \).

The first inequality has been arranged to be redundant. Hence,

\[ N_\Gamma(y) = \mathbb{R}_+ \{(2y_1, -1)\} \quad \forall (y_1, y_2) \in \text{bd} \, \Gamma \]

\( \Rightarrow \) \( (p_1, p_2 - 1) = \lambda(y)(2y_1, -1) \) with \( \lambda(y) \geq 0 \) for all \( p \) close to \( \bar{p} \) and \((x, y) \in \tilde{\Psi}(p)\).

\( \Rightarrow \) \( \lambda(y) = 1 - p_2 \geq 0 \), \( \Rightarrow \) \( y_1 = p_1/2(1 - p_2), y_2 = (p_1/2(1 - p_2))^2 \).

\( \Rightarrow \) \( \tilde{\Psi}(p) = \{(x, y) | y_1 = p_1/2(1 - p_2), y_2 = (p_1/2(1 - p_2))^2\} \) for \( p \) close to \( \bar{p} \).

Images of \( \tilde{\Psi} \) do not involve \( x \). Moreover, \( y \)-components are locally Lipschitzian functions \( \Rightarrow \) \( \tilde{\Psi} \) has Aubin property at \((\bar{p}, \bar{x}, \bar{y})\).
**Theorem (Robinson 1981)**

Let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a multifunction whose graph is a finite union of polyhedra. Then, $\Phi$ is calm at all points of its graph.

**Lemma**

$\text{gr } N_{\mathbb{R}^+}^p$ is a finite union of polyhedra.

**Proof.**

We have already shown that $\text{gr } N_{\mathbb{R}^+}^p = L^{-1}(\Lambda)$ with

$$\Lambda := \text{gr } N_{\mathbb{R}^+} \times \cdots \times \text{gr } N_{\mathbb{R}^+} \quad \text{and} \quad L(x_1, \ldots, x_p, y_1, \ldots, y_p) := (x_1, y_1, \ldots, x_p, y_p)$$

$$\text{gr } N_{\mathbb{R}^+} = \left[ \mathbb{R}^+ \times \{0\} \right] \cup \left[ \{0\} \times \mathbb{R}^- \right] \Rightarrow \Lambda = \bigcup_{(i_1, \ldots, i_p) \in \{0,1\}^p} A_{i_1} \times \cdots \times A_{i_p}$$

$$\text{gr } N_{\mathbb{R}^+}^p = \bigcup_{(i_1, \ldots, i_p) \in \{0,1\}^p} L^{-1}(A_{i_1} \times \cdots \times A_{i_p})$$

\(\square\)
Lemma

Let $\Gamma = \{ y | Cy \leq d \}$ be a polyhedron. Then, $\text{gr} \ N_\Gamma$ is a finite union of polyhedra.

Proof.

For linear mappings, the preimage formula always holds true: $N_\Gamma = C^T N_{\mathbb{R}_+^p} (Cy - d)$.

It follows that $(y, z) \in \text{gr} \ N_\Gamma \iff z \in N_\Gamma (y) \iff \exists \lambda \in N_{\mathbb{R}_+^p} (Cy - d) : z = C^T \lambda$.

Therefore, $\text{gr} \ N_\Gamma = P(\Theta)$ with

$$\Theta := \{ (y, z, \lambda) | z = C^T \lambda, \ (\lambda, Cy - d) \in \text{gr} \ N_{\mathbb{R}_+^p} \} \quad \text{and} \quad P(y, z, \lambda) := (y, z)$$

Defining $H(y, z, \lambda) := (z - C^T \lambda, \lambda, Cy - d)$ we have that $\Theta = H^{-1} \left( \{ 0 \} \times \text{gr} \ N_{\mathbb{R}_+^p} \right)$.

Thus, $\Theta$ is the preimage of a finite union of polyhedra under a linear mapping and as such is a finite union of polyhedra $A_i$: $\Theta = \bigcup_{i=1}^q A_i$. But then, $\text{gr} \ N_\Gamma = P\left( \bigcup_{i=1}^q A_i \right) = \bigcup_{i=1}^q P(A_i)$

Projection of a polyhedron is a polyhedron again. \[ \square \]
Verification via polyhedrality III

**Theorem**

If $\Gamma$ is polyhedral and $F(x, y) = Ax + By + c$ is an affine linear mapping, then the canonical perturbation mapping $\tilde{\Psi}(p) := \{(x, y) | p \in F(x, y) + N_\Gamma(y)\}$ is calm everywhere.

**Proof.**

$$\text{gr}\ \tilde{\Psi} = \{(p, x, y) | (y, p - Ax - By - c) \in \text{gr}\ N_\Gamma\} = H^{-1}(\text{gr}\ N_\Gamma)$$

Since $H$ is an affine linear mapping, and $\text{gr}\ N_\Gamma$ is a finite union of polyhedra, $\text{gr}\ \tilde{\Psi}$ is a finite union of polyhedra too. The result follows from Robinson's Theorem.
In the spot market EPEC the feasible set of the $i$th player's MPEC is given by the generalized equation:

$$F(\alpha_{-i}, \alpha_i, \beta_{-i}, \beta_i, p, t) := \left( (\alpha_{-i}, \alpha_i) + 2 \begin{bmatrix} \text{diag} (\beta_{-i}, \beta_i) \end{bmatrix} p \right)_0$$

$$\Gamma := \{(p, t) | p + Bt \geq d, \ p \geq 0, \ -\bar{t} \leq t \leq \bar{t}\}$$

(with $x$-variables and $y$-variables colored).

$\Gamma$ is polyhedral but $F$ fails to be affine linear.

It would be so if $\beta$ (quadratic coefficient) wasn’t a variable but a constant (partial bidding).

However, deviation from polyhedrality is marginal. Remedy?
Structured calmness

Theorem (Klatte/Kummer 2002)

Let $T_1 : X_1 \Rightarrow X$ and $T_2 : X_2 \Rightarrow X$ be multifunctions between metric spaces $X_1, X_2, X$. If

1. $T_1$ is calm at $(x_1, x) \in \text{gr } T_1$
2. $T_2$ is calm at $(x_2, x) \in \text{gr } T_2$
3. $T_2^{-1}$ has the Aubin property at $(x, x_2)$
4. $T_1(x_1) \cap T_2(\cdot)$ is calm at $(x_2, x)$,

then the multifunction $(T_1 \cap T_2)(x_1, x_2) := T_1(x_1) \cap T_2(x_2)$ is calm at $(x_1, x_2, x)$.

Note that intersection of calm mappings is not necessarily calm-

Hence, conditions 1. and 2. above are not sufficient.
Application to canonical perturbation mapping

Using the previous theorem, we can show the following

**Theorem (Adam/H./Outrata 2013)**

Consider the canonical perturbation mapping \( \tilde{\Psi}(p) := \{(x,y)|p \in F(x,y) + N_{\Gamma}(y)\} \).

Fix any \((\bar{x}, \bar{y}) \in \tilde{\Psi}(0)\). Assume that \( \Gamma \) is polyhedral and

\[
F(x,y) = \begin{pmatrix}
F_1(x,y) \\
F_2(y)
\end{pmatrix}
\]

with \( F_2 \) affine linear and \( \nabla F_1(\bar{x}, \bar{y}) \) surjective.

Then, \( \tilde{\Psi} \) is calm at \((0, \bar{x}, \bar{y})\).

Applies to spot market EPEC: The \( i \)th player’s MPEC is given by the generalized equation:

\[
F(\alpha_{-i}, \alpha_i, \beta_{-i}, \beta_i, p, t) := \begin{pmatrix}
F_1(\alpha_i, \beta_i, p, t) := \alpha_i + 2\beta_ip_i \\
\alpha_{-i} + 2[\text{diag } \beta_{-i}]p \\
0
\end{pmatrix}
\]

\[
\Gamma := \{(p, t) | p + Bt \geq d, \ p \geq 0 - \bar{t} \leq t \leq \bar{t}\}
\]

(\( \text{with } x\text{-variables and } y\text{-variables colored} \)).

\( \Rightarrow \Gamma \) polyhedral, \( F_2 \) affine linear and \( \nabla_{\alpha_i, \beta_i} F_1(\alpha_i, \beta_i, p, t) = (1, 2p_i) \) surjective.
Application to a class of bilevel problems

We consider the following class of bilevel problems (optimistic form):

$$\min\{\varphi(x, y) | y \in \text{argmin}\{\langle x, y^a \rangle + \delta(y^a) + \langle y^b, Cy^b \rangle + \langle c, y^b \rangle | y \in \Gamma\}\}$$  \hfill (★)

Here, $y = (y^a, y^b)$, $C$ is an arbitrary matrix of appropriate size (no rank assumption), $\delta$ is an arbitrary twice continuously differentiable function and $\Gamma$ is polyhedral.

**Theorem**

Let $(\bar{x}, \bar{y}^a, \bar{y}^b)$ be a solution to (★). Then (without any further constraint qualification):

Then, there exists some bilevel problem multipliers $v^*, u^* = (u_1^*, u_2^*)$ such that:

$$0 = \nabla_{y^a} \varphi(\bar{x}, \bar{y}) - \nabla^2 \delta(\bar{y}^a) \nabla_x \varphi(\bar{x}, \bar{y}) + u_1^*$$

$$0 = \nabla_{y^b} \varphi(\bar{x}, \bar{y}) + (C + CT)v^* + u_2^*$$

$$u^* \in D^* N_\Gamma(\bar{y}, -F(\bar{x}, \bar{y})(-\nabla_x \varphi(\bar{x}, \bar{y}), v^*))$$

All that is left in order to explicitly apply these stationarity conditions is the computation of the coderivative to the normal cone mapping of polyhedra (see below).
Proof of previous theorem

Proof.

\[
F(x, y^a, y^b) = \begin{pmatrix} F_1(x, y^a, y^b) \\ F_2(x, y^a, y^b) \end{pmatrix} = \begin{pmatrix} \nabla y^a f(x, y^a, y^b) \\ \nabla y^b f(x, y^a, y^b) \end{pmatrix} = \begin{pmatrix} x + \nabla \delta(y^a) \\ (C + C^T)y^b + c \end{pmatrix}
\]

\[\implies \nabla_x F_1(\bar{x}, \bar{y}) = I \quad \text{(surjective)} \]

\[F_2 \text{ is affine linear and } \Gamma \text{ is polyhedral. Apply previous theorem}
\]

along with abstract M-stationarity conditions for smooth MPECs. Observe that the multiplier

\[v^* = (v_1^*, v_2^*) \text{ from that theorem can be detailed by } v_1^* = -\nabla_x \varphi(\bar{x}, \bar{y}).\]
Computing the coderivative $D^* N_{\Gamma}$: Exact formulae

**Theorem (Transformation formula: Mordukhovich 2006)**

Let $C = G^{-1}(P)$, where $G \in C^2(\mathbb{R}^n, \mathbb{R}^m)$ and $P \subseteq \mathbb{R}^m$ is closed. Fix any $\bar{x} \in C$ and $\bar{v} \in N_C(\bar{x})$. If $\nabla G(\bar{x})$ is surjective, then

$$D^* N_C(\bar{x}, \bar{v})(v^*) = \left( \sum_{i=1}^m \bar{\lambda}_i \nabla^2 G_i(\bar{x}) \right) v^* + \nabla^T G(\bar{x}) D^* N_P (G(\bar{x}), \bar{\lambda}) (\nabla G(\bar{x}) v^*).$$

Here, $\bar{\lambda} = \left( \nabla G(\bar{x}) \nabla^T G(\bar{x}) \right)^{-1} \nabla G(\bar{x}) \bar{v}$.

Exact reduction of coderivative from preimage space to image space.

Application to our set $\Gamma$ described by smooth inequalities:

**Corollary**

Let $\Gamma := \{ y \in \mathbb{R}^m \mid g_i(y) \leq 0 \ (i = 1, \ldots, p) \}$, where $g_i \in C^2$. Fix any $\bar{y} \in \Gamma$ and $\bar{v} \in N_\Gamma(\bar{y})$. If $\Gamma$ satisfies LICQ at $\bar{y}$, then

$$D^* N_\Gamma(\bar{y}, \bar{v})(v^*) = \left( \sum_{i=1}^p \bar{\lambda}_i \nabla^2 g_i(\bar{y}) \right) v^* + [\nabla g(\bar{y})]^T D^* N_{\mathbb{R}_-^p} (g(\bar{y}), \bar{\lambda}) (\nabla g(\bar{y}) v^*).$$

Here, $\bar{\lambda} = \left( \nabla g(\bar{y}) [\nabla g(\bar{y})]^T \right)^{-1} \nabla g(\bar{y}) \bar{v}$. 
In exactly the same way as we computed $D^* N_{\mathbb{R}^p_+}$ in earlier exercises, we may compute $D^* N_{\mathbb{R}^p_-}$:

$$D^* N_{\mathbb{R}^p_-} (x, y)(y^*) = \emptyset \quad \text{if there exists some } i \text{ such that } \bar{x}_i = 0, \bar{y}_i > 0, y^*_i \neq 0$$

Otherwise:

$$D^* N_{\mathbb{R}^p_-} (x, y)(y^*) = \begin{cases} 
    x^*_i = 0 & \text{if } x_i = y_i = 0, y^*_i < 0 \\
    x^*_i \geq 0 & \text{if } x_i = y_i = 0, y^*_i > 0 \\
    x^*_i < 0 & \text{or } x_i < 0, y_i = 0 
\end{cases}$$
## Explicit M-stationarity conditions

**Theorem**

Let \((\bar{x}, \bar{y})\) be a local solution to the MPEC with smooth data

\[
\min \{ \varphi(x, y) | 0 \in F(x, y) + N_{\Gamma}(y) \}, \quad \Gamma := \{ y \in \mathbb{R}^p | g_i(y) \leq 0 \ (i = 1, \ldots, p) \} \quad g_i \in \mathcal{C}^2, \]

Assume that

- \(\Gamma\) satisfies LICQ at \(\bar{x}\)
- \(\bar{\Psi}(p) := \{(x, y) | p \in F(x, y) + N_{\Gamma}(y) \}\) is calm at \((0, \bar{x}, \bar{y})\).
- (without loss of generality) \(g(\bar{y}) = 0\).

Then, there exists MPEC multipliers \(u^* \in \mathbb{R}^p\) and \(v^* \in \mathbb{R}^m\) such that

\[
\begin{align*}
0 &= \nabla_x \varphi(\bar{x}, \bar{y}) + [\nabla_x F(\bar{x}, \bar{y})]^T v^* \\
0 &= \nabla_y \varphi(\bar{x}, \bar{y}) + \left( [\nabla_y F(\bar{x}, \bar{y})]^T + \sum_{i=1}^{p} \bar{\lambda}_i \nabla^2 g_i(\bar{y}) \right) v^* + [\nabla g(\bar{y})]^T u^* \\
0 &= \nabla g_i(\bar{y}) v^* \quad \forall i : \bar{\lambda}_i > 0 \\
0 &= u_i^* \quad \forall i : \bar{\lambda}_i = 0, \ \nabla g_i(\bar{y}) v^* < 0 \\
0 &\leq u_i^* \quad \forall i : \bar{\lambda}_i = 0, \ \nabla g_i(\bar{y}) v^* > 0
\end{align*}
\]

Here, \(\bar{\lambda}\) is the unique solution of \(F(\bar{x}, \bar{y}) = [\nabla g(\bar{y})]^T \bar{\lambda}\).
The case of polyhedral $\Gamma$

Under LICQ the polyhedral case comes as an immediate consequence of the previous result (without second-order terms). But an exact formula even holds without LICQ.

Let $\Gamma := \{x \in \mathbb{R}^n | Ax \leq b\}$ for some $(q, n)$-matrix $A$. Denote the rows of $A$ by $a_i$.

Fix $\bar{x} \in \Gamma$ and $\bar{v} \in N_\Gamma(\bar{x})$, i.e., $\bar{v} = A^T \lambda$ for some $\lambda \in \mathbb{R}^q_+$.

For each $x \in \Gamma$ let $I(x) := \{i | a_i x = b_i\}$. Define the family of active index sets as

$$I := \{I \subseteq \{1, \ldots, q\} | \exists x \in \Gamma : I = I(x)\}$$


$$D^* N_\Gamma (\bar{x}, \bar{v}) (v^*) = \left\{ x^* \left| (x^*, -v^*) \in \bigcup_{J \subseteq I_1 \subseteq I_2 \subseteq I(\bar{x})} P_{I_1, I_2} \times Q_{I_1, I_2} \right. \right\},$$

$$P_{I_1, I_2} = \text{con} \{ a_i | i \in \chi (I_2) \setminus I_1 \} + \text{span} \{ a_i | i \in I_1 \}$$

$$Q_{I_1, I_2} = \{ h \in \mathbb{R}^n | \langle a_i, h \rangle = 0 \ (i \in I_1), \ \langle a_i, h \rangle \leq 0 \ (i \in \chi (I_2) \setminus I_1) \}$$

and 'con' and 'span' refer to the convex conic and linear hulls, respectively. Moreover,

$$J := \{j \in I | \lambda_j > 0\} \quad \text{and} \quad \chi(I') := \bigcap \{ J \in \mathcal{I} | I' \subseteq J \} \quad \forall I' \subseteq \{1, \ldots, q\}$$

\[ \text{December 2013} \]
Computing the coderivative $D^* N_\Gamma$: Upper estimate

**Theorem (H./Outrata/Surowiec 2009)**

Let $\Gamma := \{ y \in \mathbb{R}^m | g_i(y) \leq 0 \ (i = 1, \ldots, p) \}$, where $g_i \in C^2$. Fix any $\bar{y} \in \Gamma$ and $\bar{v} \in N_\Gamma(\bar{y})$.

If $\Gamma$ satisfies MFCQ and CRCQ (Constant Rank Constraint Qualification) at $\bar{y}$, then

$$D^* N_\Gamma(\bar{y}, \bar{v})(v^*) \subseteq \bigcup_{\lambda \in N_{\mathbb{R}^p_+}} \left( \sum_{i=1}^p \bar{\lambda}_i \nabla^2 g_i(\bar{y}) v^* + [\nabla g(\bar{y})]^T D^* N_{\mathbb{R}^p_-} (g(\bar{y}), \bar{\lambda}) \right) (\nabla g(\bar{y}) v^*)$$

- Theorem was first proved by Mordukhovich/Outrata 2007 under MFCQ and calmness of the mapping
  $$\tilde{F}(p) = \{ (\lambda, y) | (\lambda, g(y)) - (p_1, p_2) \in \text{gr} N_{\mathbb{R}^p_+} \}$$

- In H./Outrata/Surowiec 2009 it was observed that the latter is implied by calmness of all mappings
  $$F_I^*(p) := \{ x \in \mathbb{R}^n | g_i(y) = p_i \ (i \in I), \ g_i(y) \leq p_i \ (i \notin I) \} \ (I \subseteq \{1, \ldots, p\})$$

- In 2011, Minchenko/Starkhovski proved that CRCQ for a system of equalities/inequalities implies calmness of the associated canonical perturbation mapping

- CRCQ for a system of inequalities is the same as CRCQ for a subsystem of equalities/inequalities.