About the mathematical foundation of Quantum Mechanics

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Lecture n° 3
15-09-2014
Lecture 3: About the future of Quantum Mechanics. Some problems and challenges

- From Quantum Mechanics to Banach algebras
- Banach algebras
- Algebraic Quantum Mechanics:
  - Algebraic Quantum Mechanics: von Neumann algebras, C*-algebras, non-associative algebras
  - Nonassociative algebras: lights and shadows. The spectrum
- About a non-associative spectral theory
- From Quantum Mechanics to Biology
- Non-associative Banach algebras in Biology
- Genetics algebras. Evolution algebras
- Evolution algebras: From Biology to Physic
- About my contribution to the theory
Let go back again to the golden year for the Science: 1932


(Mathematical foundation of the Quantum Mechanics)


(Spectral theory for hermitian operators on Hilbert spaces)


(Theory of Banach spaces)

Also in 1932, Norbert Wiener introduced the inequality

\[ \|xy\| \leq \|x\|\|y\| \]

in a normed space. Also von Neumann was interested in this. Both of them without studying further consequences of it.
Banach Algebras

- Three years later, in 1936, the Japanese mathematician Mitio Nagumo defined explicitly the notion of Banach algebra under the name of linear metric ring in


- Kôsaku Yosida also used in the same year the notion of Banach algebra under the name of metrical complete ring in the papers:

  K. Yosida, On the group embedded in the metrical complete ring, Japan J.Math.13 (1936), 7-26
  K. Yosida, On the group embedded in the metrical complete ring, Japan J.Math.13 (1936), 459,472

- In 1938, Barry Mazur proved that every complex division normed algebra is isomorphic to \( \mathbb{C} \), meanwhile every real one is isomorphic to \( \mathbb{R} \) or to the quaternions. This was an important contribution to the Gelfand theory.

- The mathematician whose work really developed the theory of Banach algebras was Israel M. Gelfand, in his dissertation thesis (1939). There, it recognized the central role of maximal ideals to derive the modern theory of Banach algebras. There results were publised in

Banach algebras

**Definition:** A Banach algebra is a Banach space $A$ together with a bilinear application $(a,b) \rightarrow ab$ (the **product**), $A \times A \rightarrow A$, such that:

i) $(ab)c = a(bc)$ (associativity of the product)

ii) $\|ab\| \leq \|a\| \|b\|$ (submultiplicative property of the norm)

**Examples:** $\mathbb{R}^n$, $\mathbb{C}^n$, $l_p$, $C[a,b]$, $L_p[a,b]$, $M_{n\times n}$, $L(X)$, $L(H)$, $C^*$-algebras

Recall the relevance of the algebra $L(H)$ in Quantum Mechanics, and the important role of the spectrum of an operator $T \in L(H)$. An abstract version of this was given:

**Definition:** An algebra $A$ has a **unit** if there exists $e \in A$ such that $ae = ea = a$, $\forall a \in A$. It is sad that $a \in A$ is **invertible** if there exists $b \in A$ such that $ab = ba = e$. If this happens, then $b$ is unique because $A$ is associative. We write $b = a^{-1}$.

**Definition:** Let $A$ be an algebra with a unit. We define the **spectrum** of $a \in A$ as $\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda e \text{ is not invertible}\}$
Banach algebras

Around the emergent Banach algebra theory, different (and some way distant) areas of mathematics were unified.

The power of this theory was clear because it provided nice and general proofs of theorems of completely different topics (holomorphic functions and spectral theory for instance). This attracted the attention of the mathematical community.

Particularly, the theory showed deep connections between the Functional Analysis, the Classical Analysis and the Spectral Theory in Physics.

The term Banach algebras seems to be coined by Warren Ambrose in this paper:


The space $L(H)$ of bounded continuous functions on a Hilbert space, essential in Quantum mechanics, was also a basic in the theory of Banach algebras. $L(H)$ non-commutative Banach algebra (with the product given by the composition of operators). Particular examples of this mathematical structure are the matrix algebras, also important in Physics.
As said before, in 1932, John von Neumann developed the mathematical foundation of Quantum Mechanics based on the Banach algebra $L(H)$. This theory explained many problems of physics, pointing out the beginning of Modern Physics. Nevertheless, in 1935, the own von Neumann pointed out his doubts about the role of this algebra.

I would like to make a confession which may seem immoral: do not believe in Hilbert space anymore…..

Now we [his coauthor F. J. Murray and himself] begin to believe that it is not the vector which matter but the lattice of all linear closed subspaces. Because

1.- The vectors ought to represent the physical states, but they do it redundantly, up to a complex factor only.

2.- And besides the states are merely a derived notion, the primitive (phenomenologically given) notion being the qualities, which correspond to the linear closed subspaces.

… But if we wish to generalize the lattice of all linear closed subspaces from a Euclidean space to infinitely many dimensions then one does not obtain a Hilbert space, but that configuration, which Murray and I called a irreducible factor (a special type of von Neumann algebra).

For this quote see:

In this way, von Neumann showed his interest in what was named Reticular Theory or Quantum Logic. Its aim is the study of the lattice $P(H)$ of the projections of an infinite dimensional Hilbert space $H$ over its linear closed susspaces (that is the logic of the quatum system).

Anyway, the needing of looking for an abstract generalization of the classical formulation of quantum mechanics, going beyond to Hilbert spaces, led to the called Algebraic Quantum Mechanic. The aim was to abstract and axiomatize the theory. The following main papers arose in this way:


Francis Joseph Murray (1911 – 1996)

Eugene Paul Wigner (1902-1995)  
Nobel Prize of Physic in 1963
Thus, the theory the \textit{von Neumann algebras} (or \textit{W *-algebras}) were developed, under the original name of \textit{rings of operators}, in the above papers of J. von Neumann and F. J. Murray (On rings of operators I, II and IV), in the paper of von Neuman (On ring on operators III) of 1940, and in others of 1938, 1943 and 1949, reprinted in the \textit{collected works of J. von Neumann} (1961), \textit{W *-algebra} was the nomenclature of Dixmier (1981) by following suggestions of Dieudonne (1957).

\textbf{Definition:} A \textbf{von Neumann algebra} is a weakly closed *-subalgebra of $L(H)$ containing the identity.

A *-subalgebra of $L(H)$ is a subalgebra closed under the adjoint operation.

A sequence $T_n$ in $L(H)$ converges weakly to $T \in L(H)$ if $\langle T_n u, v \rangle \rightarrow \langle Tu, v \rangle$, $u, v \in H$.

A C*-algebra norm closed *-subalgebra, as we will see.

But, the notion of algebra of \textit{von Neumann} was introduced by \textit{von Neumann} some years before than the quoted papers in the following work:

Algebraic Quantum Mechanics: C*-algebras

In the attempt to model the physical observables in an optimal way, after the matrix mechanics and the use of von Neumann algebras, the next idea was to consider “norm closed” subalgebras, instead of weakly closed ones.

Even more: the idea Israel Gelfand and Mark Naimark was to consider an abstract characterisation of C*-algebras, making no reference to operators on a Hilbert space. Thus the modern definition of C*-algebra arose in:

I.M. Gelfand and M. A. Naimark, On the imbedding of normed rings into the ring of operators in Hilbert space. Mat. Sbornik 12 (1943), 197-213

On the imbedding of normed rings into the ring of operators in Hilbert space

1. Gelfand (Moscow) and M. Neumark (Moscow)

§ 1. Fundamental notions

This paper is devoted to the investigation of a class of normed rings. A set $R$ is called normed ring (cf. [3]) if

(a) $R$ is a linear normed complete space in the sense of Banach [1];

(b) an (in general non-commutative) operation of multiplication is defined in $R$ with the ordinary properties ($\lambda, \mu$ are complex numbers)

Israel Gelfand
(1913-2009)

Mark A. Naimark
(1909-1978)
**Definition:** A C*-algebra is a Banach algebra $A$ which admits a conjugate linear involution $*: A \rightarrow A$ satisfying that $(ab)^* = b^*a^*$ and the C*-identity
$$\|a^*a\| = \|a\|^2 \quad (a \in A)$$  (Gelfand-Naimark axiom)

As established there the identity $\|a^*a\| = \|a\|^2$ is equivalent to the following ones:
$\|a^*a\| = \|a\|^2$ and $\|a^*\| = \|a\|$.

**Examples:** $\mathbb{R}^n, \mathbb{C}^n, M_{n\times n}, C[a, b]$ and $L(H)$, are C*-algebras

- In this paper the following theorem was proved:

**Theorem (Gelfand-Naimark):** Every arbitrary C*-algebra is isometrically *-isomorphic to a C*-algebra of bounded operators on a Hilbert space.
Inmediately the C*-algebras played a central role in the Algebraic Quantum Mechanics, thank to contributions such as the following ones (among others):


In this second work, Segal provides a series of algebraic postulates to model a quantum system. He stated that a quantum system is modeled by a C*-algebra and deduced from this the main features of the stationary states of the quantum theory.

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**POSTULATES FOR GENERAL QUANTUM MECHANICS**

**By I. E. Segal**

(Received December 3, 1946)

1. **Introduction**

We present in this paper a set of postulates for a physical system and deduce from these the main general features of the quantum theory of stationary states. Our theory is strictly operational in the sense that only the observables of the physical system are involved in the postulates. The collection of all bounded self-adjoint operators on a Hilbert space, which has previously been used as a mathematical model for the observables in quantum mechanics, satisfy the postulates, as do a variety of considerably more general mathematical structures.
Algebraic Quantum Mechanics: $C^*$-algebras

Of the existing literature the paper most closely connected with ours is that of von Neumann [7], in which another set of postulates for the observables in a physical system are given. The most notable difference between the sets of postulates is that von Neumann makes a postulate which has the effect that he can define a distributive and commutative product for any two observables. This postulate appears of have no direct physical meaning (as von Neumann remarks), and in physics the product of two observables is definable in a natural way only in the special case that they are simultaneously observable. It is therefore very desirable to dispense with the postulate, as we have done. In addition we make no assumptions as to compactness, separability, etc., of classes of observables, such as are made in [7]. Our spectral resolution is similar to a result in [7], but apart from this our results are unrelated. On the other hand, it is interesting to note that if our algebraic postulates are strengthened sufficiently, then it can be shown that the collection of observables is isomorphic (algebraically and metrically) with all self-adjoint operators in an algebra of bounded operators on a Hilbert space (the norm corresponding to the operator bound).

An aspect of our theory which is significant for general physics is that a general indeterminacy principle follows from the postulates, as the postulates are of a relatively simple character, this serves

2. Postulates and Definitions

We call a structure $\mathcal{A}$ which satisfies the following postulates a closed system of observables, or for short, a system.

Postulates
1. Algebraic postulates.
   1. $\mathcal{A}$ is a real linear space.
   2. There exists in $\mathcal{A}$ an identity element $I$ and for every $U \in \mathcal{A}$ and positive integer $n$ an element $U^n$ of $\mathcal{A}$, these being such that the usual rules for operating with polynomials in a single variable are valid: if $f, g, \text{ and } h$ are polynomials with real coefficients, and if $f(g(a)) = h(a)$ for all real $a$, then $f(g(U)) = h(U)$; here
   $$f(U) = \sum a_k U^k, \text{ if } f(a) = \sum a_k a_k.$$

   II. Metric postulates.
   There is defined for each observable $U$ a non-negative real number $\| U \|$ such that:
   1. If $a$ is an arbitrary real number and $U$ and $V$ are arbitrary elements of $\mathcal{A}$, then $\| aU \| = | a | \| U \|$, $\| U + V \| \leq \| U \| + \| V \|$.
   The vanishing of $\| U \|$ implies $U = 0$. And $\mathcal{A}$ is topologically complete when regarded as a metric space with the distance between $U$ and $V$ defined as $\| U - V \|$. (In other words, $\mathcal{A}$ is a real Banach space relative to $\| U \|$ as a norm.)
   2. $\| U^2 - V^2 \| \leq \max \{ \| U^2 \|, \| V^2 \| \}.$
   3. $\| U^2 \| = \| U \|^2$
   4. $\| \sum_{\nu \in \mathcal{S}} U^\nu \| \leq \| \sum_{\nu \in \mathcal{S}} U^\nu \|$, if $\mathcal{S} \subset \mathfrak{S}$, $\mathcal{S}$ and $\mathfrak{S}$ being finite subsets of $\mathfrak{A}$.
   5. $U^2$ is a continuous function of $U$. 
ON SEGAL’S POSTULATES FOR GENERAL QUANTUM MECHANICS

BY S. SHERMAN

(Received March 4, 1955)

1. Introduction

In [8] I. Segal has presented “a set of postulates for a physical system” and deduced from these “the main general features of the quantum theory of stationary states”. Notable precursors in this enterprise were P. Jordan, J. von Neumann, and E. P. Wigner, who in [4] studied finite dimensional, formally real, non associative algebras satisfying some additional conditions not specified here. For these “r number algebras”, which spawned the Jordan algebras studied by algebraists, [4] developed spectral theory (and thus deduced “general features sented by any algebra obtained with quasi product of real matrices.

Since the “r number algebras” being finite dimensional do not satisfy the Heisenberg commutation relations, von Neumann [7] introduced another system of postulates, which introduced topological restrictions weaker than the finite dimensional restrictions in [4]. This system of postulates includes not only the distribution postulate, but also the requirement that the set algebra be single defined.

In [4] it is stated that all the postulates except the distributive postulate arise from assumed physical conditions. In [7] the distributive postulate is regretfully introduced with the explanation that without it “an algebraic discussion will be scarcely possible”. For this new system [7] developed a spectral theory. The main model for this theory is the set of self adjoint operators of a weakly closed self adjoint algebra of operators on a real or complex Hilbert space. [7] does not solve the question of whether \( M_3^{\mathbb{A}} \) satisfies the postulates of [7]. Part 2 of [7] has not been published.
the stream of thought represented in the bibliographies of all three. The postulates of [8] (reproduced later in this paper) are much simpler than those of [7]. [8] does not assume a distributive postulate, nor is a weak topology introduced. Nevertheless a spectral theory is developed giving “the main general features of the quantum theory of stationary states”. No complete representation theory is developed. The only examples discussed in [8] are those equivalent to the set of self adjoint elements of a uniformly closed self adjoint algebra of operators on a real or complex hilbert space. The examples naturally include those mentioned before as models for [7]. It has been an unsolved problem as to whether or not these are the only models for Segal’s systems of observables. In [8] as in [7] the role of $\mathfrak{M}^S$ is not settled. It is not demonstrated whether or not $\mathfrak{M}^S$ can be normed so as to satisfy Segal’s postulates.

In view of the paucity of models for systems of observables, the prospect of establishing properties of systems of observables suggested by properties already known for self adjoint elements of a $C^*$ algebra (Gelfand and Neumark [3]) has been a tempting one. The main purpose of this paper is to destroy this prospect by presenting a class of examples of non-distributive systems of observables. These universal counterexamples not only show that Segal’s systems, which may differ in topological closure requirements, may also differ in important algebraic aspects from the system of [7]. The counterexamples also show that a number of propositions relating the order and algebraic aspects of $\mathcal{E}$, the self adjoint elements of a $C^*$ algebra, do not hold for systems of observables. In particular a system of observables may be a lattice in its natural order without being commutative [8]. It was an attempt to prove this and stronger results [2] recently established for $\mathcal{E}$ that gave rise to the counterexample. Other conjectures which are demonstrated false by the “universal counterexample” will be presented in the body of the paper.

It is also shown here that Postulate II, 4 of Segal is redundant. A result communicated to the author by A. A. Albert is used to show that $\mathfrak{M}^S$ can be normed so as to become a system of observables. Some open questions are indicated at the end of the paper.
ON A CERTAIN ALGEBRA OF QUANTUM MECHANICS

BY A. ADRIAN ALBERT
(Received November 14, 1933)

1. Introduction. P. Jordan, J. von Neumann, and E. Wigner have discussed certain linear real non-associative algebras of importance in quantum mechanics. Their algebras $\mathcal{M}$ satisfy the ordinary postulates for addition, the commutative law for multiplication, and the distributive law, but they are non-associative.

In the paper quoted above it is shown that, with a single exception, every algebra satisfying the above postulates is equivalent to an algebra $\mathcal{M}$ whose elements are ordinary real matrices $x, y, \cdots$ with products $xy$ in $\mathcal{M}$ defined by quasi-multiplication,

$$xy = \frac{1}{2}(x \cdot y + y \cdot x),$$

where $x \cdot y$ is the ordinary matrix product. This single exception is the algebra $\mathcal{M}_3^8$ of all three rowed Hermitian matrices with elements in the real non-associative algebra $C$ of Cayley numbers.

The algebras obtained by quasi-multiplication of real matrices were considered
Algebraic Quantum Mechanics: non-associative algebras

From the beginning of the mathematical formulation of Quantum Mechanics it was clear that non-associative algebras were very important models for the Quantum Mechanics.

Annals of Mathematics
Vol. 35, No. 1, January, 1934

On an Algebraic Generalization of the Quantum Mechanical Formalism

By P. Jordan, J. v. Neumann, and E. Wigner
(Received November 10, 1933)

Introduction

One of us has shown that the statistical properties of the measurements of a quantum mechanical system assume their simplest form when expressed in terms of a certain hypercomplex algebra which is commutative but not associative. This algebra differs from the non-commutative but associative matrix algebra usually considered in that one is concerned with the commutative expression \( \frac{1}{2}(A \times B + B \times A) \) instead of the associative product \( A \times B \) of two matrices. It was conjectured that the laws of this commutative algebra would form a suitable starting point for a generalization of the present quantum mechanical theory. The need of such a generalization arises from the (probably) fundamental difficulties resulting when one attempts to apply quantum mechanics to questions in relativistic and nuclear phenomena.

The mathematical foundations underlying this generalization have already

It was P. Jordan who showed that the statistical properties of the measurement of a quantum mechanical system where modeled by a non-associative algebra in:

P. Jordan, Uber die multiplikation quanten-mechanischer grossen, Zschr. f. Phys. 80, (1933); 285-291
We recall that:

**Definition:** a Banach algebra is a Banach space $A$ endowed with a product, that is a bilinear map $(a, b) \rightarrow ab$, satisfying for every $a, b, c \in A$ that:

i) **Associativity:** $(ab)c = a(bc)$

ii) **Submultiplicativity of the norm:** $\|ab\| \leq \|a\| \|b\|$ (continuity of the product).

Why is natural or reasonable to require the associativity of the product?

**Non-associative algebras** has being considered from the very beginning of the theory of algebras. Indeed if we consider a base in a linear space and we define a multiplication table with them then we obtain an algebra which is likely is non-associative. Moreover, even in the eventual case of obtaining an associative algebra: Try to prove the associativity of the product!

**Example:** Take $A = \text{Lin}\{u_1, u_2\}$ and define the following multiplication table:

<table>
<thead>
<tr>
<th></th>
<th>$u_1$</th>
<th>$u_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_1$</td>
<td>0</td>
<td>$u_1$</td>
</tr>
<tr>
<td>$u_2$</td>
<td>$u_2$</td>
<td>$u_2$</td>
</tr>
</tbody>
</table>

Since $(u_1u_2)u_1 = 0 \neq u_1(u_2u_1) = u_1$ we obtain a non-associative algebra (which is a non-associative Banach algebra under any algebra norm).

We propose to delete (i) in the above definition! Therefore,

**Definition:** a non-associative Banach algebra is a Banach space $A$ endowed with a product, that is a bilinear map $(a, b) \rightarrow ab$, that $\|ab\| \leq \|a\| \|b\|$, for every $a, b, c \in A$. 
The scientific interest of nonassociative algebras lies in the origins of modern mathematics for many reasons.

One of them is that the relationship between the complex plane and the Euclidean Geometry was a motivation to look for generalizations field of complex numbers to study the associated geometries.

Thus, hypercomplex numbers arose, and the study of normed division algebras was on fashion from the end of the nineteenth century. In this way, a nonassociative normed division algebra was born: the octonions (or Cayley numbers).

The octonions (or Cayley numbers) were created by T. Graves in 1843 (according to a note published in 1847 by William R. Hamilton Trans. Royal Irish Acad.). Germinal ideas in works of C.F Degen (1818).

The first fundamental work about the octonions is due to P. Jordan, with the next paper which add another reference to our “gold” year 1932.

New ideas: Replace the complex numbers by the octonions, order to explain certain properties of space-charge for elementary particles. Thus non-associative models also burst in Quantum Mechanics, formally.

- More precisely, in 1933, the Jordan algebras arose to formalize in a better way the notion of observable in Quantum Mechanics:


- John von Neumann himself acknowledges this progress and participates in them. In fact a few months later, a first joint work with Jordan appeared:


- In this way, the so-called nonassociative quantum mechanic began to take shape. Nowadays, it takes into account observables quantities as well as unobservable ones (related to quarks). To integrate unobservable quantities in the mathematical formulation of quantum mechanics it is necessary to enlarge the algebra $L(H)$ to see it within a convenient algebra which has to be non-associative.
A non-associative quantum mechanics is proposed in which the product of three and more operators can be non-associative. The multiplication rules of the octonions define the multiplication rules of the corresponding operators with quantum corrections. The self-consistency of the operator algebra is proved for the product of three operators. Some properties of the non-associative quantum mechanics are considered. It is proposed that some generalization of the non-associative algebra of quantum operators can be helpful for understanding of the algebra of field operators with a strong interaction.

6. CONCLUSIONS AND DISCUSSION

In this notice it is shown that one can try to destroy the octonions multiplication rules to receive a non-associative quantum mechanics which is not equivalent to the standard quantum mechanics. The distinction is that the algebra of operators is generated not only by the canonical commutation/anti-commutations relationships but also by the rules regulating the brackets order in the operators product. The construction of such quantum mechanics is very complicated problem and now we would like to list some unresolved problems:

1. In this notice we show the self-consistency of the product of three operators. But the self-consistency of the product of four and more operators is an open problem.

2. Non-associative operators can not have the representation on the functions like \( \hat{p}_x = -i\hbar \frac{\partial}{\partial x} \) in the standard quantum mechanics. In this context the question is how one can define eigenstates and eigenfunctions and what are observables of the non-associative operators?
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3. Probably such non-associative quantum mechanics is hardly realized in the Nature but one can presuppose that some generalization of the non-associative quantum mechanics for the quantum field theory can exist. The matter is that only for free fields the algebra of quantized fields is known. The algebra of strongly interacting quantum fields (in some regimes of quantum chromodynamics, quantum gravity and so on) remains unknown. It means that the algebra of operators of the strongly interacting fields is much more complicated the algebra of operators generating only by the canonical commutation/anti-commutation relationships and probably the algebra of strongly interacting fields can be non-associative one.

4. One can presuppose that the formulation of the rules for the rearrangement of brackets in some non-associative operators product will lead to the appearance of a second Planck constant.

REFERENCES
Nowadays, no doubt about the interest of nonasociative algebras in Physics. See for instance:


In spite of the interest of non-associative algebras in Physics and other sciences: There is no a spectral theory for general non-associative algebras. There has been no a clear idea even about what we understand by the inverse of an element. Therefore the spectrum was not defined in a non-associative context.

We give an answer to this problem: we establish a notion of spectrum of an element in a non-associative algebra. With this spectrum we will extend many important results of the classical theory of Banach algebras to non-associative framework.

J. C. Marcos y M.V.V. The multiplicative spectrum and the uniqueness of the complete norm topology, FILOMAT 28 (2014), 473-485
Nonassociative algebras: lights and shadows


“One way of trying to create new mathematics..., is to generalize the theory by dropping or weakening some of its hypotheses. If we play this axiomatic game with the concept of an associative algebra, we are likely to be led to the concept of a non-associative algebra, which is obtained simply by dropping the associative law of multiplication. ...If this stage is reached, it is quite certain that one would soon abandon the project, since there is very little of interest that can be said about non-associative algebras in general”.

Personal contribution:
Paradoxically, the associativity is a superfluous requirement in the theory of the so-called Radical of Jacobson.

Also it is superfluous in the theory of automatic continuity of surjective homomorphisms from a Banach algebra onto a semi-simple Banach algebra.

Nonassociative algebras: the spectrum

How we got it? Remember that non-associative division algebras were studied by Albert

**Remark:** Let $A$ be an associative complex algebra with a unit, $e$, and let $a \in A$. 
There exists $a^{-1} \in A$ such that $aa^{-1} = e = a^{-1}a$ if and only if $L_a$ and $R_a$ are bijective.

Indeed, if this is the case, then $L_a^{-1} = a^{-1}L_a$ and $R_a^{-1} = R_a^{-1}$.

We recall that $L_a(x) = ax$ and $R_a(x) = xa$.

**Reformulation:** If $A$ is a complex algebra with a unit then

$$
\sigma(a) = \{ \lambda \in \mathbb{C} : a - \lambda e \text{ is not invertible} \} = \{ \lambda \in \mathbb{C} : L_a - \lambda I \text{ is not bijective} \} \cup \{ \lambda \in \mathbb{C} : R_a - \lambda I \text{ no es biyectivo} \}
$$

This is:

$$
\sigma^A(a) = \sigma^L(A)(L_a) \cup \sigma^L(A)(R_a)
$$

**Observation:** $L(X)$ is an associative algebra, even if $X = A$ with $A$ non-associative

**Idea:** FILOMAT 28 (2014), 473-485.

**Definition:** Let $A$ be a complex non-associative algebra with a unit. We define the spectrum of $a \in A$ as

$$
\sigma^A(a) = \sigma^L(A)(L_a) \cup \sigma^L(A)(R_a)
$$
About a non-associative spectral theory

This definition is extended to the real case in the standard way (that is as in the associative setting). Similarly for algebras without a unit.

**With this notion of spectrum many important classical results of the theory of Banach algebras are generalized to the non-associative framework.** Particularly, this is the case of the Jacobson’s radical theory.

**Proposition:** Let $A$ be a non-associative Banach algebra and let $\sigma^A(a)$ be the spectrum of $a \in A$. Then $\sigma^A(a)$ is a non-empty compact subset of $\mathbb{C}$. Indeed, $|\lambda| \leq \|a\|$, for every $\lambda \in \sigma^A(a)$

**Definition:** The spectral radius of $a \in A$ is defined by $r(a) = \sup\{|\lambda| : \lambda \in \sigma^A(a)\}$

**Proposition (spectral radius formula):** Let $A$ be a non-associative Banach algebra and let $a \in A$. Then $r(a) = \max\{\|L_a^n\|^{1/n}, \|R_a^n\|^{1/n}\}$

All these results generalize the corresponding associative ones.

**What is next? What to do now?**
To explore the relevance of this ideas in Physics, as well as in other branch of the Science.
The non-associative structures have been used in the Science from the most diverse environments: Genetics for instance.

The mathematical study of the genetic inheritance began in 1856, with the works of Gregor Mendel himself. In fact he was a pionner in using mathematical notation to express the genetics laws.

After Mendel, other researchers such as Jennings (1917), Serebrovskij (1934) and Glivenko (1936) provided a more precise mathematical approach of Mendel's laws.

This culminated in the algebraic formulation of Mendel's laws in the famous papers:


After this, many authors have made relevant contributions to Genetics, in the context of non-associative algebras: Schafer, Gonshor, Haldane, Holgate, Heuch, Reiersöl, Abraham, Lyubich and Wörz Busekros, among others.

Thus, it seems that non-associative algebras are the appropriate mathematical framework for studying Mendelian genetics.
Non-associative Banach algebras in Biology

- Mathematical formulation of the **Second law of Mendel**:

\[
\begin{array}{c|ccc}
 & NN & Ng & gg \\
\hline
NN & NN & \frac{1}{2}NN + \frac{1}{2}Ng & Ng \\
Ng & \frac{1}{2}NN + \frac{1}{2}Ng & \frac{1}{4}NN + \frac{1}{2}Ng + \frac{1}{4}gg & \frac{1}{2}Ng + \frac{1}{2}gg \\
gg & Ng & \frac{1}{2}Ng + \frac{1}{2}gg & gg \\
\end{array}
\]

*Gráfico que detalla la segunda ley de Mendel o ley de segregación equitativa.*

_Idea:_ This is a non-associative Banach algebra
Genetics algebras. Evolution algebras

There are so many non-associative algebras that have attracted the interest of geneticists, that it is not feasible to make an exhaustive list of them. Serve as an example the following classes of algebras: Mendelian, gametic, zygotic, baric, train algebras, Bernstein algebras or evolution algebras. All of them are called genetic algebras.

J. P. Tian, (2008): During the early days in this area, it appeared that the general genetic algebras or broadly defined genetic algebras, could be developed into a field of independent mathematical interest, because these algebras are in general not associative and do not belong to any of the well-known classes of nonassociative algebras such as Lie algebras, alternative algebras, or Jordan algebras.... They possess some distinguishing properties that lead to many interesting mathematical results.


Outstanding genetic algebras are the evolution algebras. They are defined as those algebras generated by a base \( \{u_1, \ldots, u_n\} \) provided with a multiplication table such that \( u_i u_j = 0, \) \( i \neq j. \)
After the works of E. Baur and C. Correns, it was clear that many hereditary mechanisms do not follow the Mendel's laws. This is the case of systems polygenes or multiple alleles, or the asexual inheritance. Therefore, the non-Mendelian genetics arose. Nowadays, the Molecular Genetics is based on it.

Evolution algebras are basic mathematical structures that model the non-Mendelian genetics. As pointed out already, they are unclassifiable non-associative Banach algebras (they are not Jordan, nor Lie. Even they are not power associative algebras).

Evolution algebras has applications in many branch of Mathematics. For instance:

Tian (2008): “an evolution algebra defined by a Markov chain is a Markov evolution algebra. In other words, properties of Markov chains can be revealed by studying their evolution algebras. Moreover, Markov chains, as a type of dynamical systems, have a hidden algebraic aspect”.

Moreover they also have applications in Physics.
Evolution algebras also modelize some topics in Physics. For instance: This Feynmann diagram define an evolution algebra.

In theoretical physics, the Feynman diagrams are mathematical expressions governing the behavior of subatomic particles. The scheme was introduced in 1948 by Richard Feynman. The interaction of sub-atomic particles can be difficult to understand intuitively, and these diagrams allow for a simple visualization of it.

As David Kaiser writes, "Feynman diagrams have revolutionized nearly every aspect of theoretical physics". (Winner of the 2007 Pfizer Prize from the History of Science Society).

While the Feynman diagrams are applied primarily to Quantum Field Theory, they can also be used in other fields, such as Solid-State Theory.
About my contribution to the theory


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- J. C. Marcos. Álgebras no asociativas normadas completas. Teoría Espectral. (Tesis 2011)
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Advised joint with Mercedes Siles
Thank you very much for your attention!