Introduction to spectral theory of unbounded operators.

H. Najar

Dep. Mathematics. Taibah University
Dep. Mathematics. F.S. Monastir University. Tunisia
Laboratoire de recherche: Algèbre Géométrie et Théorie Spectrale : LR11ES53.
Email: hatemnajar@yahoo.fr

Definition

A Banach space $\mathcal{A}$ is a **Banach algebra** if there exists a multiplication on $\mathcal{A}$ such that $\mathcal{A}$ is an algebra with

1. $\forall x, y \in \mathcal{A} : \| xy \| \leq \| x \| \cdot \| y \|.$

2. It has an identity $e \in \mathcal{A}$ i.e $\forall x \in \mathcal{A}; \ x = ex = xe$, suppose that $\| e \| = 1$.

A Banach algebra is a Banach $\star$-algebra, (**algebra**) if there exists an involution $f : \mathcal{A} \to \mathcal{A}$ $\forall x, y \in \mathcal{A}, \alpha \in \mathbb{C}, \ f(x + y) = f(x) + f(y), \ f(xy) = f(y)f(x), \ f(\alpha x) = \bar{\alpha}f(x), \text{ and } f^2(x) = x.$
**Definition**

A *algebra is called a $\mathbb{C}^*$-algebra if we have

$$\forall x \in \mathcal{A}, \| f(x)x \| = \| x^*x \| = \| x \|^2.$$  \hspace{1cm} (1)

**Remark**

*Equation (1), says $\mathbb{C}^*$-identity, is equivalent to*

$$\forall x \in \mathcal{A}; \| x^* \| = \| x \|.$$
Example:

1. For any space $X$, the bounded linear operators $\mathcal{B}(X)$, form a Banach algebra with identity $1_X$.

2. For any Hilbert space $\mathcal{H}$, $\mathcal{B}(\mathcal{H})$ is a $\mathbb{C}^*$-algebra when it is equipped with the adjoint map

$$\ast : H \in \mathcal{B}(\mathcal{H}) \mapsto H^* \in \mathcal{B}(\mathcal{H})$$
Duality

If $X$ and $Y$ are normed linear spaces and $T : X \to Y$, then we get a natural map $T^* : Y^* \to X^*$ by

$$T^* f(x) = f(Tx), \quad \forall f \in Y^*, x \in X.$$ 

In particular, if $T \in B(X, Y)$, then $T^* \in B(Y^*, X^*)$. In fact,

$$\|T^*\|_{B(Y^*, X^*)} = \|T\|_{B(X,Y)}.$$

To prove this, note that

$$|T^* f(x)| = |f(Tx)| \leq \|f\| \cdot \|T\| \cdot \|x\|.$$ 

Therefore $\|T^* f\| \leq \|f\| \cdot \|T\|$, so $T^*$ is indeed bounded, with

$$\|T^*\| \leq \|T\|.$$
Also, given any \( y \in Y \), we can find \( g \in Y^* \) such that \( |g(y)| = \|y\| \), \( \|g\| = 1 \). Applying this with \( y = T x \) (\( x \in X \) arbitrary), gives

\[
\|T x\| = |g(T x)| = |T^* g x| \leq \|T^*\| \cdot \|g\| \cdot \|x\| = \|T^*\| \|x\|.
\]

This shows that

\[
\|T\| \leq \|T^*\|.
\]

Note that if \( T \in B(X, Y) \), \( U \in B(Y, Z) \), then

\[
(UT)^* = T^* U^*.
\]
Let $X$, $Y$ be Hilbert spaces. Let $T \in \mathcal{B}(X, Y)$ be a bounded linear transformation.

$$\| T \| = \sup \{ \| Ah \|_Y : \| h \|_X \leq 1 \}.$$

Then the norm of $T$ satisfies:

$$\| T \|^2 = \| T^* \|^2 = \| T^* T \|$$

where $T^*$ denotes the adjoint of $T$. Indeed Let $h \in X$ such that $\| h \|_X \leq 1$. Then:

$$\| Th \|_Y^2 = \langle Ah, Ah \rangle_Y = \langle T^* Th, h \rangle_X \leq \| T^* Th \|_X \| h \|_X (Cauchy – Schwarz Inequality) \leq \| T^* T \| \| h \|_X^2 \leq \| T^* \| \| T \| \ $$
it follows that

\[ \| T \| ^2 \leq \| T^* T \| \leq \| T^* \| \| X \|. \]

That is,

\[ \| T \| \leq \| T^* \|. \]

By substituting \( T^* \) for \( T \), and using \( T^{**} = T \) from [Double Adjoint is Itself], the reverse inequality is obtained. Hence

\[ \| T \| ^2 = \| T^* T \| = \| T^* \|^2 . \]
Examples

**Example 1:** For any compact Hausdorff space $S$;

$$\mathcal{C}(S) = \{ f : S \rightarrow \mathbb{C} | f \text{ continuous} \},$$

equipped with the norm

$$\| f \|_{\infty} = \sup_{x \in S} | f(x) |$$

is a commutative Banach algebra with identity $f = 1$, the involution

$$f^*(x) \equiv \overline{f(x)}$$

transforms it on a $\mathbb{C}^*$-algebra.
Example 2: The analytic functions

$$f : D^1 = \{z \in \mathbb{C}; |z| < 1\} \rightarrow \mathbb{C}$$

with norm

$$\| f \|_{\infty} = \sup_{z \in D} |f(z)|,$$

the involution: $$f(z) \mapsto \overline{f(z)}$$

form a commutative Banach algebra, but not a $$\mathbb{C}^*$$-algebra. With $$f(z) = e^{iz}$$; we have

$$\| f \|_{\infty}^2 = e^2 \neq \| f^* f \|_{\infty} = 1.$$
Definition

For a Banach algebra $\mathcal{A}$ with identity $1_\mathcal{A}$ we define

1. The resolvent set

$$\varrho_\mathcal{A}(x) = \{\lambda \in \mathbb{C} | x - \lambda 1_\mathcal{A} \text{ Has two sided bounded inverse}\}$$

2. The spectrum of $x \in \mathcal{A}$

$$\sigma_\mathcal{A}(x) = \mathbb{C} \setminus \varrho_\mathcal{A}(x).$$

3. We call the inverse of $x - \lambda 1_\mathcal{A}$, the resolvent and denote as

$$R_\lambda(x) = (x - \lambda 1_\mathcal{A}) = \frac{1}{x - \lambda 1_\mathcal{A}}.$$
First resolvent formula

Lemma

For any $\lambda, \nu \in \varrho(x)$.

\[
R_\lambda(x) - R_\nu(x) = (\lambda - \nu)R_\lambda(x)R_\nu(x) = (\lambda - \nu)R_\nu(x)R_\lambda(x).
\]

Proof: Multiply both sides with $x - \lambda \mathbb{1}_A$ or $x - \nu \mathbb{1}_A$. 

H. Najar
Introduction to spectral theory of unbounded operators.
Theorem

Let $A$ be a Banach algebra with identity and $x, y \in A$ with $x$ invertible and $\|x^{-1}y\| < 1$, then $x - y$ is invertible,

$$(x - y)^{-1} = \sum_{n=0}^{\infty} (x^{-1}y)^n x^{-1},$$

the series being absolutely convergent and

$$\|(x - y)^{-1}\| \leq \|x^{-1}\|/(1 - \|x^{-1}y\|).$$
Proof:

\[ \| \sum_{n=0}^{\infty} (x^{-1}y)^n x^{-1} \| \leq \|x^{-1}\| \sum_{n=0}^{\infty} \|x^{-1}y\|^n \]
\[ \leq \|x^{-1}\|/(1 - \|x^{-1}y\|), \]

so the sum converges absolutely and the norm bound holds. Also

\[ \sum_{n=0}^{\infty} (x^{-1}y)^n x^{-1}(x - y) = \sum_{n=0}^{\infty} (x^{-1}y)^n - \sum_{n=0}^{\infty} (x^{-1}y)^{n+1} = 1X, \]

and similarly for the product in the reverse order.
Remark

If $f$ is an analytic function, i.e. $f$ can be represented by a convergent power series, $f(x) = \sum_{n=0}^{\infty} a_n x^n$, we can define $f(T) = \sum_{n=0}^{\infty} a_n T^n$ (which is defined since $B(X)$ is Banach).
Proposition

Let $X$ be a Banach space, $T \in B(X)$ with $\|T\| < 1$. Then $(I - T)^{-1} \in B(X)$ and $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$ (the Neumann series) in $B(X)$.

proof Let $S_k = \sum_{n=0}^{k} T^n$. Then, for $k < \ell$,

$$\|S_\ell - S_k\| = \left\| \sum_{k < n \leq \ell} T^n \right\| \leq \sum_{k < n \leq \ell} \|T^n\| \leq \sum_{k < n \leq \ell} \|T\|^n \leq \sum_{n=k+1}^{\infty} \|T\|^n \xrightarrow{k \to \infty} 0$$

Hence, $\{S_k\}$ is Cauchy in $B(X)$, so convergent. Let $S = \lim_{k \to \infty} S_k$ in $B(X)$. 

H. Najar
Introduction to spectral theory of unbounded operators.
\[
(I - T) S_k x = \sum_{n=0}^{k} (T^n - T^{n+1}) x = x - T^{k+1} x \xrightarrow{k \to \infty} x
\]

since \( \| T^{k+1} x \| \leq \| T \|^{k+1} \| x \| \). On the other hand \((I - T) S_k x \to (I - T) S x\) as \(k \to \infty\). Hence,

\[
S = (I - T)^{-1}.
\]
Proposition

Let $T \in B(X)$. Then $\rho(T) \subseteq \mathbb{C}$ is an open set, i.e. $\sigma(T) = \mathbb{C} \setminus \rho(T)$ is closed, and the resolvent function

$$\varrho(T) \ni \lambda \mapsto R_\lambda(T) \in B(X)$$

is a complex analytic map from $\rho(T)$ to $B(X)$ with

$$\|R_\lambda(T)\| \leq \frac{1}{d(\lambda, \sigma(T))},$$

i.e. for all $\lambda_0 \in \rho(T)$, there exists $r > 0$ such that

$$R_\lambda(T) = \sum_{n=0}^{\infty} a_n (\lambda - \lambda_0)^n T^n$$

for all $\lambda \in B_r(\lambda_0)$. 
Proof: Use that \((I - T)^{-1} = \sum_{n=0}^{\infty} T^n\) if \(\|T\| < 1\) and

\[
T - (\lambda - \mu)I = (T - \lambda I)(I - \mu R_\lambda(T)) = (T - \lambda I)S(\mu).
\]

Then \(S(\mu)\) is invertible if \(|\mu| \|R_\lambda(T)\| < 1\). Hence,

\[
R_{\lambda - \mu}(T) = S(\mu)^{-1}R_\lambda(T) = \sum_{k=0}^{\infty} \mu^k R_\lambda(T)^{k+1}.
\]
Proposition

Let $X, Y$ be Banach spaces. Then the set of invertible operators in $B(X, Y)$ is an open set. If $X \neq 0$ and $Y \neq 0$, then for $S, T \in B(X)$, $T$ invertible and $\|S - T\| < \|T^{-1}\|^{-1}$ implies $S$ is invertible.

proof: Let $R = T - S$. Then $S = T(I - T^{-1}R) = (I - RT^{-1})T$ where $\|T^{-1}R\| < 1$ and $\|RT^{-1}\| < 1$. 
Important implication of Neumann series Theorem

1. \( \{ x \in \mathcal{A} | 0 \in \varrho(x) \} \) is open.
2. \( \forall x \in \mathcal{A}; \varrho(x) \) is an open subset of \( \mathbb{C} \), so \( \sigma(x) \) is a closed set.
3. \( \forall x \in \mathcal{A}, \) the resolvent

\[
\lambda \mapsto R_\lambda(x) = (x - \lambda 1_\mathcal{A})^{-1}
\]

is an \( \mathcal{A} \)-valued analytic function. In particular

\[
\lim_{\lambda \to \lambda_0} \frac{R_\lambda(x) - R_{\lambda_0}(x)}{\lambda - \lambda_0} = R_{\lambda_0}^2(x).
\]

4. \( \forall f \in \mathcal{A}^* : \varrho(x) \ni \lambda \mapsto f(R_\lambda(x)) \in \mathbb{C} \) is analytic.

4. \( \forall x \in \mathcal{A}, \sigma(A) \neq \emptyset \) and it is a compact subset of the disc of radius \( \| x \| \).
Definition

Let $\Omega$ be an open set of $\mathbb{C}$, and $\mathcal{A}$ is a Banach space. Let $f : \Omega \rightarrow \mathcal{A}$. We say that $f$ is analytic in $\Omega$ if for any $\lambda_0 \in \Omega$

$$\lim_{\lambda \to \lambda_0} \frac{f(\lambda) - f(\lambda_0)}{\lambda - \lambda_0} = f'(\lambda_0),$$

exists. It is equivalent to $\varphi \circ f : \mathbb{C} \rightarrow \mathbb{C}$ is analytic for any $\varphi \in \mathcal{A}'$. 
Suppose that $\sigma(x) = \emptyset$, so $\rho(x) = \mathbb{C}$, we conclude that $R_\lambda(x)$ is an entire function with value in $\mathcal{A}$. For $|\lambda| > \|x\|$, 

$$R_\lambda(x) = -\sum_{n=0}^{+\infty} \frac{x^n}{\lambda^{n+1}}.$$ 

So 

$$\|R_\lambda(x)\| \leq \frac{1}{|\lambda| - \|x\|}.$$ 

$R_\lambda(x)$ is a bounded and entire function, so by Liouville theorem, we deduce that $R_\lambda(x)$ is constant on $\mathbb{C}$. As 

$$\lim_{|\lambda| \to \infty} R_\lambda(x) = 0.$$ 

We get $R_\lambda(x) = 0, \forall \lambda \in \mathbb{C}$, which is absurd.
**Remark**

*The fact that the spectrum of an element of $A$ is non empty it is a generalization of the fact that any matrix of $M_n(\mathbb{C})$ has at least one eigenvalue.*
Spectral radius formula

Theorem

∀x ∈ A

1. \( \lim_{n \to +\infty} \| x^n \|^{1/n} \) exists and equal \( r(x) \).

2. \( r(x) = \sup\{ |\lambda| \mid \lambda \in \sigma(x) \} \).
Remark

An element of an algebra $A$ is invertible or not is a property which is purely algebraic. So the spectrum and the spectral radius of $x$ depend only on the algebraic structure of $A$ and not of the metric or the topology, but the limit in the last theorem depends on the properties of the metric of $A$. It is one of the remarkable aspects of the theorem, which affirms the correspondence of two quantities with different origins.
Remark

The algebra $A$ could be a subalgebra of another Banach algebra $B$. So it is possible for an $x \in A$ to be non invertible in $A$ and invertible in $B$. So the spectrum of $x$ depends on the algebra. If we note by $\sigma_A(x)$ (resp. $\sigma_B(x)$) the spectrum of $x$ relatively to $A$ (resp. $B$), so $\sigma_A(x) \subset \sigma_B(x)$. The spectral radius is the same in $A$ and $B$. 

Proof:

1. Set $a_n = Ln\|x^n\|$, then

$$\forall n, m \in \mathbb{N}; a_{n+m} \leq a_n + a_m.$$ 

Fix $k \in \mathbb{N}$ and write $n = mk + r; 0 \leq r \leq k - 1$

$$a_n \leq ma_k + \max_{0 \leq r \leq k-1} a_r, \Rightarrow \limsup_{n \to \infty} \frac{a_n}{n} \leq \frac{a_k}{k}$$

$$\Rightarrow \limsup_{n \to \infty} \frac{a_n}{n} \leq \inf_{k} \frac{a_k}{k} \leq \liminf_{n \to -\infty} \frac{a_n}{n}.$$ 

2. Let $\alpha$ be the limit of $\|x^n\|^{\frac{1}{n}}$. Let $\lambda \in \sigma(x)$, so $\lambda^n \in \sigma(x^n)$ so

$$|\lambda^n| \leq \|x^n\|.$$ 

We get that $r(x) = \sup_{\lambda \in \sigma(x)} |\lambda| \leq \alpha$. 

H. Najar

Introduction to spectral theory of unbounded operators.
The opposite inequality is based on the theory of holomorphic functions and entire series. Let $\Omega = D(0, \frac{1}{r(x)})$, if $r(x) = 0, \Omega = \mathbb{C}$. Consider $f : \Omega \to A$ defined $f(0) = 0$ and

$$f(\lambda) = R_{1/\lambda}(x), \quad \lambda \in \Omega \setminus \{0\}.$$  

Using the properties of the resolvent we can write that for $0 < |\lambda| < \frac{1}{\|x\|}$

$$f(\lambda) = -\sum_{n=0}^{+\infty} \lambda^{n+1} x^n.$$  

Let $R$ be the radius of convergence of the power series $R \geq d(0, \Omega^c) = \frac{1}{r(x)}$. Using Hadamard formula

$$\frac{1}{R} = \limsup_{n \to +\infty} \|x^n\|^{\frac{1}{n}},$$
So finally

$$\limsup_{n \to +\infty} \| x^n \| \frac{1}{n} \leq r(x).$$
Application: Volterra Integral Kernels

Let $K : [0, 1] \times [0, 1] \to \mathbb{C}$, continuous $V_K : C([0, 1]) \to C([0, 1])$

$$f \mapsto \int_0^t K(t, s)f(s)ds.$$ 

We have $\| V_K \|_\infty \leq \| K \|_\infty \| f \|_\infty$ So $V_K \in B(C([0, 1]))$.

$$(V^n_K f)(t) = \int_{0 \leq s_1 \leq \cdots \leq s_n \leq t} K(t, s_n)K(s_n, s_{n-1}) \cdots K(s_2, s_1)f(s_1)ds_1 \cdots ds_n.$$ 

$$\| V^n_K f \|_\infty \leq \| K \|_\infty^n \| f \|_\infty \cdot \sup_{t \in [0,1]} Vol\{ (s_1, \cdots, s_n) \mid 0 \leq s_1 \leq \cdots \leq t \} \leq \| K \|_\infty^n / n! \cdot \| f \|_\infty.$$ 

So $r(V_K) = \lim_{n \to \infty} \| V^n_K \|_1^{1/n} \leq \lim_{n \to \infty} \| K \|_\infty \| K \|_\infty^{1/n} = 0$. and $\sigma(V_K) = \{0\}$. (Hint: $Ln(n!) \approx nLn(n)$)
How we can define $f(x)$ for a large class of functions $f$ and (un)bounded linear operator $x$?

1. Polynomial functional calculus.
3. Continuous functional calculus.
Let \( \mathcal{A} \) be a Banach algebra with identity and
\[
P(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_0, \quad a_j \in \mathbb{C} \text{ a polynomial.}
\]
If \( x \in \mathcal{A} \), then
\[
P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \in \mathcal{A}.
\]

**Spectral mapping**

**Theorem**

\[
\forall x \in \mathcal{A} : \quad \sigma(P(x)) = P(\sigma(x)) = \{ P(\lambda) \in \mathbb{C} ; \lambda \in \sigma(x) \}.
\]
Lemma

Let \( x_1, \cdots, x_n \in A \) be mutually committing, then

\[
y = x_1 \cdots x_n \text{ invertible} \iff x_1, \cdots, x_n \text{ are each invertible}
\]

Proof:

1. \[ \Rightarrow x_1(x_2 \cdots x_n)y^{-1} = yy^{-1} = 1_A \]
   \[
y^{-1}(x_1 \cdot x_2 \cdots x_n) = 1_A = y^{-1}(x_2 \cdots x_n)x_1 = y^{-1}(x_1 \cdots x_n) = y^{-1}y = 1_A \]
   So \( x_1 \) has left and right inverses. So it is invertible and are the same.

2. \[ \Leftarrow y^{-1} = x_n^{-1} \cdots x_1^{-1}. \]
Proof of the spectral mapping

Let

$$\lambda \in \sigma(P(x)) \iff q(x) = P(x) - \lambda \text{ is not invertible.}$$

$$Q(t) = (t - \mu_1) \cdots (t - \mu_n).$$ As \( x - \mu_i \) and \( x - \mu_j \) commute for any \( i, j \), applying the last Lemma we get

$$\lambda \in \sigma(P(x)) \iff \exists j, \mu_j \in \sigma(x) \iff \lambda \in P(\sigma(x)).$$
Let \( f : \mathbb{C} \to \mathbb{C} \), an entire function \( f(t) = \sum_{n=0}^{\infty} a_n t^n \). For \( x \in \mathcal{A} \),

\[
f(x) = \sum_{n=0}^{\infty} a_n x^n \in \mathcal{A}.
\]

(2)

More general function \( f : B(0, r) \to \mathbb{C} \) analytic with \( r > r(x) \). Using Cauchy integral formula we write

\[
f(x) = \frac{1}{2\pi i} \oint_{|\lambda|=r} f(\lambda)(\lambda - x)^{-1} d\lambda
\]
Definition

Let $x \in \mathcal{A}$ and $G \subset \mathbb{C}$ open connected domain such that $\sigma(x) \subset G$. Let $f : G \to \mathbb{C}$ analytic and $\Gamma \subset G \cap \varrho(x)$ a contour. We set

$$f(x) = \frac{1}{2\pi i} \oint_{\Gamma} f(\lambda)(\lambda - x)^{-1} d\lambda \in \mathcal{A}.$$  \hspace{1cm} (3)
Proposition

The equation (3) define an application from the algebra of analytic functions on $G \supset \sigma(x)$ to $A$. This map is linear and satisfies for $f, g : G \to \mathbb{C}$ analytic on $G \supset \sigma(x)$ and $\Gamma_f, \Gamma_g$ admissible contours s.t $\Gamma_f \cap \Gamma_g = \emptyset$.

$$f(x)g(x) = (fg)(x).$$

$$f(x) = \frac{1}{2\pi i} \oint_{\Gamma_f} f(\lambda)(\lambda-x)^{-1} d\lambda, \quad g(x) = \frac{1}{2\pi i} \oint_{\Gamma_g} g(\mu)(\mu-x)^{-1} d\mu.$$
\[
\begin{align*}
\text{f}(x)g(x) &= \frac{1}{(2\pi i)^2} \oint_{\Gamma_f} \oint_{\Gamma_g} f(\lambda)g(\mu)(\lambda - x)^{-1}(\mu - x)^{-1} d\lambda d\mu \\
&= \frac{1}{(2\pi i)^2} \oint_{\Gamma_g} \left( \oint_{\Gamma_f} \frac{1}{\lambda - \mu} f(\lambda) d\lambda \right) g(\mu)(\mu - x)^{-1} d\mu \\
&\quad - \frac{1}{(2\pi i)^2} \oint_{\Gamma_f} \left( \oint_{\Gamma_g} \frac{1}{\lambda - \mu} g(\mu) d\mu \right) f(\lambda)(\lambda - x)^{-1} d\lambda \\
&= \frac{1}{2\pi i} \oint_{\Gamma_f} (\lambda - x)^{-1} g(\lambda) f(\lambda) d\lambda = (fg)(x).
\end{align*}
\]

So we get an algebraic homomorphism.
Theorem

∀x ∈ A and analytic f : G → C on domain G ⊃ σ(x).

\[ \sigma(f(x)) = f(\sigma(x)) = \{ f(\lambda) | \lambda \in \sigma(x) \} . \]
Proof

If $\mu \notin f(\sigma(x))$, then $G \ni \lambda \to g(\lambda) = (f(\lambda) - \mu)^{-1}$ is analytic. So $g(x)$ is the inverse of $f(x) - \mu$, so $\mu \notin \sigma(f(x))$. If $\mu \in f(\sigma(x))$; then $\exists \lambda \in \sigma(x); \mu = f(\lambda)$. Then

$$g(z) = \frac{f(z) - f(\lambda)}{z - \lambda},$$

has a false singularity at $z = \lambda$. Hence is analytic on $G$. So,

$$f(x) - \mu = (x - \lambda)g(x) = g(x)(x - \lambda).$$

So $f(x) - \mu$ is not invertible since $\lambda \in \sigma(x)$ i.e $\mu \in \sigma(f(x))$. 

H. Najar

Introduction to spectral theory of unbounded operators.
Some particular elements of a $\ast$-algebra $A$

**Definition**

1. $x \in A$ is normal iff $x^*x = xx^*$.
2. $x \in A$ is self-adjoint iff $x^* = x$.
3. $x \in A$ is positive iff $\exists y \in A; x = yy^*$.
4. $x \in A$ is projection iff $x^2 = x = x^*$.
5. $x \in A$ is unitary iff $x^*x = xx^* = 1_A$. 
Introduction to spectral theory of unbounded operators.
Theorem

If \( x \) is normal in \( \mathbb{C}^* \)-algebra \( \mathcal{A} \), then \( r(x) = \| x \| \).

Proof:

\[
\| x^2 \| = \| xx^* \| = \| x \|^2.
\]

By induction \( n \in \mathbb{N}^* \),

\[
\| x^{2n} \| = \| x \|^{2n}.
\]

So,

\[
r(x) = \lim_{n \to \infty} \| x^{2n} \|^{1/2n} = \| x \|.
\]
Remark

The norm of a $\mathbb{C}^*$-algebra $A$ is uniquely determined by the algebraic structure

$$\| x \|^2 = \| xx^* \| = r(xx^*) = \sup \{| \lambda | ; \lambda \in \sigma(xx^*) \}.$$
**Theorem**

Let \( x \in \mathcal{A} \).

1. If \( x \) is unitary, then \( \sigma(x) \subset \partial \mathbb{D} \).
2. If \( x \) is self-adjoint, then \( \sigma(x) \subset \mathbb{R} \).

**Proof:**

Let \( x \in \mathcal{A} \) be an unitary operator, then

\[
\| x \|^2 = \| xx^* \| = \| 1_{\mathcal{A}} \| = 1.
\]

As \( x^{-1} = x^* \), then \( 0 \notin \sigma(x) \),

\[
x^{-1} - \lambda^{-1} = x^{-1} \lambda^{-1} (\lambda - x) \forall \lambda \neq 0,
\]

we conclude that

\[
\lambda \in \sigma(x) \iff \lambda^{-1} \in \sigma(x^{-1}).
\]
Let $y = e^{ix} = \sum_{n=0}^{\infty} \frac{i^n}{n!} x^n$. As the involution $\ast$ is a continuous map on $\mathcal{A}$, then $y^\ast = e^{-ix}$, and

$$y^\ast y = yy^\ast = 1_{\mathcal{A}}.$$

So $y$ is unitary operator and

$$\sigma(y) \subset \partial \mathbb{D}$$

and

$$\sigma(y) = e^{i\sigma(x)} \subset \partial \mathbb{D} \iff \sigma(x) \subset \mathbb{R}.$$
In the following we consider $X, \ Y$ two Hilbert spaces and linear operator $A : \mathcal{D}(A) \subset X \rightarrow Y$. We suppose that $\mathcal{D}(A)$ is dense in $X$.

**Examples:** Maximal multiplication operator associated with measurable $f : M \rightarrow \mathbb{C}$ over some measure space $(M, \mu)$

$$\mathcal{D}(M_f) = \{ \psi \in L^2(M, \mu) | M_f \psi = f \psi, \psi \in L^2(M, \mu) \}.$$  

**Lemma**

Suppose that $(M, \mu)$ is σ-finite. Then we have equivalence

1. $M_f \in B(L^2(M, \mu))$
2. $f \in L^\infty(M, \mu)$
Proof: ” $\iff$ ”, for all $\psi \in \mathcal{D}(M_f)$:

$$\|M_f\psi\| = \left( \int |f\psi|^2 d\mu \right)^{\frac{1}{2}} \leq \|f\|_{\infty} \cdot \|\psi\|.$$  

” $\Rightarrow$ ” As $(M, \mu)$ is $\sigma$-finite, $\exists (M_n)_n$:

$$M = \bigcup_n M_n, \mu(M_n) < \infty.$$  

Suppose that

$$\|M_f\| = \sup\{\|M_f\psi\| | \psi \in \mathcal{D}(M_f), \|\psi\| = 1\} < \infty.$$  

Consider $\chi_{n,A} = \chi\{x \in M_n, |f(x)| > A\}, A \in [0, \infty).$
\[ A^2 \cdot \mu\{x \in M_n | |f(x)| > A\} \leq \int |f|^2 |\chi_n, A| d\mu \leq \|M_f\|^2 \mu\{x \in M_n | |f(x)| > A\}. \]

This gives that

\[ \mu\{x \in M_n | |f(x)| > A\} = 0, \text{ when } A > \|M_f\|, \forall n. \]

\[ \Rightarrow f \in L^\infty(M, \mu). \text{ Thus } M_f \text{ is an unbounded operator with } D(M_f) \neq L^2(M, \mu) \text{ in case } f \notin L^\infty(M, \mu). \]
Differential operator on $I = (0, 1)$

$$T_0 : C^1 \to L^2(I), \quad T_0 \psi = -i \psi'$$

$$f_n(x) = x^n, n \in \mathbb{N}, \quad T_0 f_n(x) = -inx^{n-1}, \quad \|T_0 f_n\| = \frac{n\sqrt{2n + 1}}{\sqrt{2(n - 1)}}.$$

$$T_{max} : W^{1,2}(I) \to L^2(I), \quad T_{max} \psi = -i \psi'$$

Here

$$W^{1,2}(I) = \{ \psi : I \to \mathbb{C} | \psi, \psi' \in L^2(I) \}.$$ 

It is an Hilbert space when equipped by the norm

$$\|\psi\|^2_{W^{1,2}} = \|\psi\|^2_{L^2} + \|\psi'\|^2_{L^2}.$$

Both operators are unbounded. \( T_{max} \) is an extension of \( T_0 \).
Definition

Let $B : \mathcal{D}(B) \to Y$ and $A : \mathcal{D}(A) \to Y$. We say that $A$ is an **extension** of $B$, if $\mathcal{D}(B) \subset \mathcal{D}(A)$ and $Ax = Bx$ for all $x \in \mathcal{B}$, we write

$$B \subset A.$$
Closed and closable operators

**Definition**

Let $A : \mathcal{D}(A) \to Y$ be a linear operator on Hilbert spaces $X, Y$ with $\mathcal{D}(A)$ is dense in $X$

1. We call the **graph** of $A$ the set

   $$\text{Grph}(A) = \{(x, Ax) \in X \times Y; x \in \mathcal{D}(A)\},$$

   and the graph norm of $x \in \mathcal{D}(A)$ is $\|x\|_A = \|(x, Ax)\|_{X \times Y}.$

2. $A$ is said to be **closed** if $\text{Graph}(A)$ is a closed subset of $X \times Y$, with respect to the topology induced by $\|(x, y)\|_{X \times Y}^2 = \|x\|_X + \|y\|_Y.$

3. We call $A$ is **closable** if it has a closed extension. We denote the smallest closed extension of $A$ by $\overline{A}.$
Remark

\( X \times Y \) is an Hilbert space with the scalar product

\[ \langle (x, y), (x', y') \rangle_{X \times Y} = \langle x, x' \rangle_X + \langle y, y' \rangle_Y. \]
**Lemma**

$G \subseteq X \times Y$ is a graph of an operator $A : \mathcal{D}(A) \to Y$ if and only if $G$ is a subspace with the property:

$$(0, y) \in G \Rightarrow y = 0.$$
Proof: $\iff$ Let $(x, y), (x, y') \in G$ as $G$ is a subspace we get that $(0, y - y') \in G \implies y = y'$ so for every $x \in X$, there is at most one $y \in Y$ such that $(x, y) \in G$. So the map $A : D(A) \to Y$ with

$$D = \{x \in X | \exists y \in Y : (x, y) \in G\},$$

we set

$$Ax = y.$$

It is a well defined as linear operator with $\text{Graf}(A) = G$. 

H. Najar
Introduction to spectral theory of unbounded operators.
Lemma

Let \((A, \mathcal{D}(A))\) be a linear operator. \(A\) is closable if and only if \(\text{Graph}(A)\) is a graph.

Proof: \(\iff\) Let \(B : \mathcal{D}(B) \to Y\), with

\[
\mathcal{D}(B) = \{x \in X; \exists y \in Y : (x, y) \in \text{Graph}(A)\},
\]

we define

\[Bx = y.\]

It is a linear operator with \(\text{Graph}(B) = \overline{\text{Graph}(A)}\) and \(\text{Graph}(A) \subset \text{Graph}(B)\), and hence \(\mathcal{D}(A) \subset \mathcal{D}(B)\).
Let $B : \mathcal{D}(B) \to Y$ be a closed extension of $A$. If $(0, y) \in \text{Graph}(A)$, then $(0, y) \in \text{Graph}(B)$; i.e $y = 0$. 
Characterization of closed operators

Theorem

For a linear operator $A : \mathcal{D}(A) \to Y$ densely defined on $\mathcal{D}(A) \subset X$ the following properties are equivalent:

1. $A$ is closed.
2. $(\mathcal{D}(A), \| \cdot \|_A)$ is complete.
3. If $(x_n)_n \subset \mathcal{D}(A)$ with $x_n$ converges to $x$ and $Ax_n$ converges to $y$ then $x \in \mathcal{D}(A)$ and $Ax = y$. 

H. Najar

Introduction to spectral theory of unbounded operators.
**Proof:** (1) ⇒ (3) Let \((x_n) \subset D(A)\) with \(x_n\) converges to \(x\) and \(Ax_n\) converges to \(y\). Then \((x_n, Ax_n) \in Graph(A)\), with

\[
\| (x_n, Ax_n) - (x, y) \|_{X \times Y} \to 0.
\]

Thus \((x, y) \in \overline{Graph(A)} = Graph(A)\), i.e. \(x \in D(A)\) and \(Ax = y\).

(3) ⇒ (2) Let \((x_n) \subset D(A)\) be a Cauchy sequence w.r.t. \(\| \cdot \|_A\). Then \((x_n)\) is a Cauchy sequence w.r.t. \(\| \cdot \|_X\) and \((Ax_n)\) is a Cauchy sequence w.r.t. \(\| \cdot \|_Y\). Completeness of \(X\) and \(Y\) imply:

\[
\exists x \in X, y \in Y \text{ such that }
\| x_n - x \|_X \to 0, \| Ax_n - y \|_Y \to 0.
\]

Thus \(x \in D(A)\) and \(y = Ax\) and

\[
\| (x_n, Ax_n) - (x, y) \|_{X \times Y} \to 0.
\]
(2) \(\Rightarrow\) (1) Let \((x_n, Ax_n) \in Graph(A)\) converges to \((x, y)\). Then \((x_n) \subset D(A)\) is a Cauchy sequence w.r.t. \(\|\cdot\|_A\), and hence

\[
\exists x' \in D(A) : \|x' - x_n\|_A \to 0, x_n \to x', Ax_n = Ax'.
\]

Uniqueness of the limit in \(X\) and \(Y\) yields that \(x = x'\) and \(Ax' = y\).
Example 1: Dirac Delta function on 
\( X = L^2((-1, 1)), \mathcal{D}(A) = C((-1, 1)) \),

\[
(A\psi)(x) = \psi(0).
\]

This operator is not closable as there exists \((\psi_n) \subset C((-1, 1))\) with \(\psi_n(0) = 1\) and \(\|\psi_n\| \to 0\) and \(A\psi_n = 1 \neq 0\).
Differentiation operators on $I \subset \mathbb{R}$

Example 1:

\[
T_{\text{max}} : W^{1,2}(I) \to L^2(I), \quad T_{\text{max}} \psi = -i \psi'
\]

Here

\[
W^{1,2}(I) = \{ \psi : I \to \mathbb{C} | \psi, \psi' \in L^2(I) \}.
\]

$T_{\text{max}}$ is closed since $\| \cdot \|_{T_{\text{max}}} = \| \cdot \|_{W^{1,2}}$ and $W^{1,2}(I)$ is a Hilbert space with norm $\| \cdot \|_{W^{1,2}}$.

\[
T_0 : C^1 \to L^2(I)
\]

$T_0$ is closable.
Remark

The closure $\overline{A}$ of a closable operator $A : \mathcal{D}(A) \to Y$ is uniquely defined through

$$\mathcal{D}(\overline{A}) = \{ x \in X | \exists (x_n) \subset \mathcal{D}(A) : x_n \to x; (Ax_n) \text{ converges} \}$$

$$\overline{A}x = \lim_{n \to \infty} Ax_n.$$
Definition

Let $A : \mathcal{D}(A) \subset X \to Y$ be a **densely** defined linear operator on Hilbert spaces $X, Y$. The operator $A^* : \mathcal{D}(A^*) \to X$, with

$$\mathcal{D}(A^*) = \{y \in Y|\exists y^* \in X : \langle Ax, y \rangle_Y = \langle x, y^* \rangle_X, \forall x \in \mathcal{D}(A)\},$$

$$A^*y = y^*,$$

is called the adjoint of $A$. 
Example: Differential operators $T_0$ and $T_{max}$

Let $\psi \in C_c^\infty(I)$ and $\varphi \in W^{1,2}(I)$. Then,

$$\langle T_0 \psi, \varphi \rangle = \int_a^b -i \psi(x) \cdot \overline{\varphi(x)} \, dx = \left[ -i \psi(x) \cdot \overline{\varphi(x)} \right]_a^b$$

$$= \int_a^b \psi(x) \cdot -i \varphi'(x) \, dx = \langle \psi, T_{max} \varphi \rangle.$$  \hspace{1cm} (4) (5)

Thus

$$T_0^* = T_{max}.$$
It is possible to describe the adjoint using the graph. Let

\[ J : X \times Y \rightarrow Y \times X \]

\[(x, y) \mapsto J((x, y)) = (−y, x)\]

\(J\) is an isometric isomorphism.
Lemma

Let $A : \mathcal{D}(A) \to Y$, $B : \mathcal{D}(B) \to Y$ be two operators densely defined on $X$.

1. $\text{Graph}(A^*) = (J\text{Graph}(A))^\perp = J(\text{Graph}(A)^\perp)$.
2. $B \subset A \Rightarrow A^* \subset B^*$. 
Proof:
(1) By definition of $A^*$

$$
\text{Graph} \ (A^*) = \{(y, z) \in Y \times X | \langle Ax, y \rangle = \langle x, z \rangle, \forall x \in D(A)\}
$$

$$
= \{(y, z) \in Y \times X | \langle (-Ax, x), (y, z) \rangle_{Y \times X} = 0, \forall x \in D(A)\}
$$

$$
= \{(y, z) \in Y \times X | \langle J(v, w), (y, z) \rangle_{Y \times X} = 0, \forall v, w \in \text{Graph}(A)\}
$$

$$
= \left( J(\text{Graph}(A)) \right)^\perp = J(\left( \text{Graph}(A) \right)^\perp).
$$
(2)

\[ \text{Graph}(B) \subset \text{Graph}(A) \implies J(\text{Graph}(B)) \subset J(\text{Graph}(A)) \]
\[ \implies (J(\text{Graph}(A)))^\perp \subset (J(\text{Graph}(B)))^\perp \]
\[ \implies \text{Graph}(B^*) \supset \text{Graph}(A^*). \]
Let $A : \mathcal{D}(A) \to Y$, $\mathcal{D}(A) \subset X$ be a densely defined operator on Hilbert spaces $X$, $Y$. Then,

1. $A^*$ is closed.
2. If $A$ admits a closure $\overline{A}$, then $\overline{A}^* = A^*$.
3. $A^*$ is densely defined if and only if $A$ is closable.
4. If $A$ is closable, then it is closure $\overline{A}$ is $(A^*)^*$. 

H. Najar
Introduction to spectral theory of unbounded operators.
**Proof:**

(1) Since $V^\perp$ is closed for any $V$, the graph $\text{Graph}(A^*)$ is closed by previous lemma.

(2)

$$\text{Graph}(A^*) = \left( J(\text{Graph}(A)) \right)^\perp = (J(\overline{\text{Graph}(A)}))^\perp = J(\overline{\text{Graph}(A)}) = \overline{J(\text{Graph}(A))} = \text{Graph}(A^*).$$
(3) We have

\[
\overline{\text{Graph}(A)} = \overline{(\text{Graph}(A)^\perp)^\perp} = \overline{(J^{-1}(\text{Graph}(A^*)))^\perp} \quad \text{(by the precedent lemma)}
\]

\[
= \{(x, y) \in X \times Y \mid \langle J^{-1}(z, A^*z), (x, y) \rangle_{X \times Y} = 0, \quad \forall z \in \mathcal{D}(A^*) \}
\]

Thus \((0, y) \in \overline{\text{Graph}(A)} \iff y \in \mathcal{D}(A^*)^\perp.\)

\[
\overline{\text{Graph}(A)} \quad \text{is a graph} \iff \mathcal{D}(A^*) \quad \text{is dense.}
\]
(4) Using (3), we conclude that $A^{**}$ is well defined and

$$\text{Graph}(A^{**}) = (J^{-1}(\text{Graph}(A^*)))^\perp = (J^{-1}J(\text{Graph}(A)^\perp))^\perp = (\text{Graph}(A)^\perp)^\perp = \overline{\text{Graph}(A)}, \ i.e \ \overline{A} = A^{**}.$$
**Definition**

A densely defined linear operator $A : \mathcal{D}(A) \to X, \mathcal{D}(A) \subset X$ on Hilbert space $X$ is called

1. Symmetric iff $A \subset A^*$.
3. Essentially self adjoint iff $A^*$ is self adjoint.

**Remark**

*If $A$ is essentially self-adjoint operator then $A \subset A^{**} = A^*$ i.e $A$ is symmetric.*
Theorem

1. Every symmetric operator $A$ is closable with $\overline{A} \subset A^*$

2. Equivalent statements

   1. $A$ is e.s.a. ($A^{**} = A^*$).
   2. $\overline{A} = A^*$.
   3. $\overline{A}$ is self-adjoint, in this case $\overline{A}$ is the unique self adjoint extension of $A$. 

H. Najar

Introduction to spectral theory of unbounded operators.
Proof:

(1) $A^*$ is closed extension of $A$.
(2) $(1) \Rightarrow (2)$, $\overline{A} = A^{**} = A^*$
(2) $\Rightarrow (3)$: $\overline{A} = A^{**} = \overline{A}^*$
(3) $\Rightarrow (1)$ $A^* = \overline{A}^* = \overline{A} = A^{**}$ (last therem)

If $\tilde{A}$ is a s.a. extension of $A$, then $\tilde{A} = \tilde{A}^* \subset A^* = \overline{A} \subset \tilde{A} \Rightarrow \tilde{A} = \overline{A}$. 
Example: Maximal multiplication operator, with measurable $f : M \rightarrow \mathbb{R}$ over some $\sigma$-finite measure space $(M, \mu)$

$$D(M_f) = \{ \psi \in L^2(M, \mu) \mid f\psi L^2(M, \mu) \}$$

$$M_f\psi = f\psi.$$

Let $(x, y) \in Graph(M_f^*)$, $y = M_f^*x$, then for $\psi \in D(M_f)$.

$$| \int x f\psi d\mu | \leq \| y \| \cdot \| \psi \|,$$

so $\psi \mapsto \int x f\psi d\mu = \langle M_f\psi, x \rangle = \langle \psi, y \rangle$ extends uniquely to a bounded functional on $L^2(M, \mu)$, i.e $f\overline{x} \in L^2(M, \mu)$ and $\overline{y} = f\overline{x}$.

Therefore $(x, y) \in Graph(M_f^*) \iff y \in D(M_f)$ and $y = f\overline{x}$. So

$$M_f = M_f^*.$$
Particular case: \( M = \mathbb{R}^d, \mu = \text{Lebesgue measure} f(k) = |k|^2 \), define a self-adjoint operator \( M_f \). The Fourier transformation

\[
\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)
\]

\[
(\mathcal{F}\psi)(x) = \int_{\mathbb{R}^d} e^{-ik \cdot x} \psi(k) \frac{dk}{(2\pi)^{d/2}}.
\]

Define a unitary transformation with

\[
\mathcal{F} M_f \psi = -\Delta \mathcal{F} \psi,
\]

and \( \mathcal{F}(\mathcal{D}(M_f)) = \mathcal{D}(L) \), the Laplacian

\[
\mathcal{D}(L) = \{ \psi \in L^2(\mathbb{R}^d) | \Delta \psi \in L^2(\mathbb{R}^d) \} = \mathcal{F}\mathcal{D}(M_f).
\]

\[
L \psi = -\Delta \psi.
\]
Remark

Using similar reasoning allows to conclude that all differential operators of the form $\text{Pol}(\nabla)$ are self-adjoint provided $\text{Pol}(ik) \in \mathbb{R}$ for all $k \in \mathbb{R}^d$. 
We recall that, $T_0 \psi = -i \psi'$, $\mathcal{D}(T_0) = \mathcal{C}_c^\infty(I)$, and
$\mathcal{D}(T_{\text{max}}) = W^{1,2}(I)$. $T_0 \subset T_{\text{max}} = T_0^*$. So $T_0$ is a symmetric operator

$$\mathcal{D}(\overline{T_0}) = \{ \psi \in W^{1,2}(I) | \psi(a) = \psi(b) = 0 \} = W_0^{1,2}(I).$$

$$\overline{T_0} \psi = -i \psi',$$

it is not essentially self adjoint.
For $\beta \in [0, 2\pi)$ let
\[
D(T_\beta) = \{ \psi \in W^{1,2}(I) | \psi(b) = e^{i\beta} \psi(a) \}.
\]

Then
\[
T_\beta \psi = -i \psi'.
\]

Then
1. $T_0 \subset T_\beta \subset T_{\text{max}}$
2. $T_\beta \subset T_\beta^*$ i.e $T_\beta$ symmetric, since $\varphi, \psi \in D(T_\beta)$

\[
\beta \psi, \varphi \rangle = \int_a^b -i\psi'(x)\varphi(x)dx
\]

\[
= \left[ -i\psi(x)\varphi(x) \right]^b_a + \int_a^b \psi(x) - i\varphi'(x)dx
\]

\[
= \langle \psi, T_\beta \varphi \rangle.
\]
\[ T^*_\beta = T_\beta, \text{ since } \forall \varphi \in \mathcal{D}(T^*_\beta), \psi \in \mathcal{D}(T_\beta) \]

\[
\int_{a}^{b} -i\psi'(x)\varphi(x) \, dx = \langle T_\beta \psi, \varphi \rangle = \langle \psi, T^*_\beta \varphi \rangle \quad (9)
\]

\[
= \int_{a}^{b} \psi(x) - i\varphi'(x) \, dx. \quad (10)
\]

So \( T_0 \) has infinitely many self adjoint extensions.
Theorem

For any densely defined linear operator $A$ on a Hilbert space $X$.

$$\overline{\text{Ran}A} \oplus \ker A^* = X.$$
Proof: It suffices to prove that $\ker A^*$ is the orthogonal complement of $\text{Ran} A$. Let $u \in \text{Ran} A$ and $v \in \ker A^*$. Then there exists $f \in D(A)$ such that $u = Af$. We compute

$$\langle u, v \rangle = \langle Af, v \rangle = \langle f, A^* v \rangle = 0.$$  

and thus $\ker A^* \subseteq (\text{Ran} A)^\perp$. Now let $w \in (\text{Ran} A)^\perp$. For $u = Af \in \text{Ran} A$, we have

$$0 = \langle u, w \rangle = \langle Af, w \rangle = \langle f, A^* w \rangle, \quad \forall f \in D(A).$$

(Notice that $\langle Af, w \rangle = 0$ implies that $w \in D(A^*)$.) As $D(A)$ is dense, it follows that $A^* w = 0$, that is $(\text{Ran} A)^\perp \subseteq \ker A^*$.  

H. Najar
Introduction to spectral theory of unbounded operators.
Let $A : \mathcal{D} \rightarrow X$ be a symmetric operator with the property that $\text{Ran}(A) = X$. Then $A$ is selfadjoint.

**Proof:** As $\mathcal{D}(A) \subset \mathcal{D}(A^*)$, it suffice to show that if $f \in \mathcal{D}(A^*)$ then $f \in \mathcal{D}(A)$.

Let $g = A^* f$. As $\text{rang}(A) = X$, there exists $h \in \mathcal{D}(A)$ so that $g = Ah$.

$$\forall v \in \mathcal{D}(A), \quad \langle Av, f \rangle = \langle v, A^* f \rangle = \langle v, g \rangle = \langle v, Ah \rangle = \langle Av, h \rangle.$$ 

If $u \in X$ is arbitrary, there exists $v \in \mathcal{D}(A) = X$ such that $u = Av$. Hence we have

$$\langle u, f \rangle = \langle u, h \rangle \quad \forall u \in X.$$ 

So $f = h \in \mathcal{D}(A)$. 

H. Najar

Introduction to spectral theory of unbounded operators.
Let $T$ be a symmetric operator, the following assertions are equivalents

1. $T$ is self-adjoint.
2. $T$ is closed and $\ker(T^* \pm i) = \{0\}$.
3. $\text{Ran}(T \pm i) = X$. 
(1) ⇒ (2): Let \( T \) is self-adjoint and \( \varphi \in D(T^*) = D(T) \) such that \( \varphi \in \text{Ker}(T^* \pm i) \). So
\[
\mp i\langle \varphi, \varphi \rangle = \langle \mp i\varphi, \varphi \rangle = \langle T^*\varphi, \varphi \rangle = \langle T\varphi, \varphi \rangle = \langle \varphi, T^*\varphi \rangle = \pm i\langle \varphi, \varphi \rangle.
\]
So \( \varphi = 0 \).

(2) ⇒ (3): Let \( y \in \text{Ran}(T \pm i)^\perp \), then \( \langle (T \pm i)x, y \rangle = 0 \) for any \( x \in D(T) \). So \( y \in D(T^*) \) and \( T^*y = \pm iy \). So
\( y \in \text{Ker}(T^* \mp i) = \{0\} \) so \( \text{Ran}(T \pm i) \) dense in \( X \). Let’s prove that \( \text{Ran}(T \pm i) \) is closed. Indeed for all \( x \in D(T) \).
\[
\| (T \pm i)x \|_2^2 = \| Tx \|_2^2 + \| x \|_2^2,
\]
as \( T \) is symmetric. This yields that if \( x_n \in D(T) \) a sequence such that \( (T \pm i)x_n \to y \), so \( x_n \) converges to \( x \). As \( T \) is closed we deduce that \( x \in D(T) \) and \( (T \pm i)x = y \). So \( y \in \text{Ran}(T \pm i) \) so \( \text{Ran}(T \pm i) = X \).
(3) ⇒ (1) Let $x \in D(T^*)$, as $Ran(T \pm i) = X$ there exists $y \in D(T)$ such that $(T - i)y = (T^* - i)x$. As $T \subset T^*$, we have $x - y \in D(T^*)$ and $(T^* - i)(x - y) = 0$. So

$$x - y \in ker(T^* - i) = Ran(T + i)^\perp = X^\perp = \{0\}.$$ 

So $x = y \in D(T)$ and $D(T) = D(T^*)$.
Example 1: Let $X = l^2(\mathbb{N})$, let $A$ be the operator with domain

$$\mathcal{D}(A) = \{ x = (x_n)_{n \in \mathbb{N}} : x_n \neq 0, \text{for finitely many } n \}$$

and

$$Ax := \left( \sum_{i=1}^{\infty} x_i, 0, 0, 0, \cdots \right).$$

Let’s determine $A^*$. Let $e_n$ be the standard unit vector. Pick $y \in \mathcal{D}(A^*)$, then

$$1 \cdot \overline{y}_1 = \langle Ae_n, y \rangle = \langle e_n, A^* y \rangle = 1 \cdot \overline{(A^* y)_n}, \forall n \in \mathbb{N},$$

this yields that $A^* y = 0$, and we obtain $y_1 = 0$. So for any $y \in \mathcal{D}(A^*)$ we have $y_1 = 0$ and $A^* y = 0$. 

H. Najar
Introduction to spectral theory of unbounded operators.
Now consider the linear operator $B$ given by

$$\mathcal{D}(B) = \{(y_n)_n \in l^2(\mathbb{N}) : y_1 = 0\}, \text{By} = 0.$$ 

Let $y \in \mathcal{D}(B)$,

$$\langle Ax, y \rangle = \langle x, By \rangle \forall x \in \mathcal{D}(A).$$

There for, $y \in \mathcal{D}(A^*)$ and $A^*y = By$. So

$$\mathcal{D}(A^*) = \{(y_n)_n \in l^2(\mathbb{N}) : y_1 = 0\}; A^*y = 0, \forall y \in \mathcal{D}(A^*).$$

Since $\mathcal{D}(A^*)$ is not dense in $l^2$, the operator $A$ is not closable.
Example 2: Let $X = L^2([0, 1])$, $\mathcal{D}(T_0) = C_\infty^\infty((0, 1))$

$$T_0 f = -f''.$$ 

By integration by part we see that $T_0$ is symmetric. Let’s compute for $f \in \mathcal{D}(T_0)$,

$$\langle T_0 f , 1 \rangle = - \int_0^1 f'' 1 = [-f' 1]_0^1 + \int_0^1 f' 1' = 0.$$ 

So $1 \in \mathcal{D}(T^*)$ and moreover $T^*1 = 0$. So $(1, 0) \in Graph(T_0^*)$. For any $x \in [0, 1]$, and $f \in \mathcal{D}$ we have

$$|f(x)| = \left| \int_0^x \int_0^1 f''(s)dsdt \right| \leq \int_0^x \int_0^1 |f''(s)|dsdt$$

$$\leq \int_0^1 \int_0^1 |f''(s)|dsdt = \int_0^1 |f''(s)|ds \leq \|T_0 f\|.$$
In particular if \( \| T_0 f \| \leq \frac{1}{2}; \) then \( |f(x)| \leq \frac{1}{2}, \forall x \in [0, 1], \) so \( |-f(x)| \geq \frac{1}{2} \) and \( \| 1 - f \| \geq \frac{1}{2}. \) So in all cases we have

\[
\| 1 - f \|^2 + \| T_0 f - 0 \|^2 \geq \frac{1}{4}.
\]

So \( (1, 0) \notin \overline{\text{Graph}T_0} \) and \( T \) is not essentially self-adjoint.

\( T_0^* f = -f'' \), with

\[
\mathcal{D}(T_0^*) = H^2([0, 1]) = \{ f \in C^1([0, 1]) : f'' \in L^2([0, 1]) \}.
\]
Definition

Let $A : \mathcal{D}(A) : \rightarrow X, \mathcal{D}(A) \subset X$ be a closed linear operator in some Hilbert space $X$. Then

$$\rho(A) = \{ \lambda \in \mathbb{C} | A - \lambda \text{ has a bounded inverse} \}.$$ 

Is the resolvent set and

$$\sigma(A) = \mathbb{C}\setminus \rho(A),$$

the spectrum of $A$ and $R_\lambda(A) = (A - \lambda)^{-1}$ is the inverse of $A - \lambda$. 
The spectrum of a closed linear operator $A : \mathcal{D}(A) \to X, \mathcal{D}(A) \subset X$, decomposes into the following components

1. $\sigma_p(A) = \{ \lambda \in \mathbb{C} | \ker(A\lambda) \neq \{0\} \}$. It is called the point spectrum or set of eigenvalues of $A$. Every $x \in \ker(A - \lambda) \setminus \{0\}$ is called eigenvectors of $A$ with eigenvalue $\lambda \in \sigma_p(A)$.

2. $\sigma_r(A) = \{ \lambda \in \mathbb{C} | \ker(A - \lambda) = \{0\}, \overline{\text{Range}(A - \lambda)} \neq X \}$. Is called the residual spectrum of $A$.

3. $\sigma_c(A) = \{ \lambda \in \mathbb{C} | \ker(A - \lambda) = \{0\}; \text{range}(A - \lambda) \neq X; \text{Range}(A - \lambda) = X \}$. Is called continuous spectrum.
For any closed operator $A : \mathcal{D}(A) \subset X \to X$, we have the following disjoint decomposition

$$\sigma(A) = \sigma_p(A) \cup \sigma_r(A) \cup \sigma_c(A).$$
Others decomposition of the spectrum exists in case $A = A^*$.

1. Lebesgue decomposition $\sigma_{pp}(A) = \sigma_{p}(A)$ pure point spectrum,
   $$\sigma_c(A) = \sigma_{sc}(A) \cup \sigma_{ac}(A).$$

2. $\sigma_{disc}(A) = \{\lambda; \text{isolated eigenvalue of } A \text{ with finite multiplicity}\}$
   $$\sigma_{ess}(A) = \sigma(A) \setminus \sigma_{disc}(A).$$
Example: $M_f : L^2(M, \mu) \to L^2(M, \mu)$. Let $\lambda \in \mathbb{C}$, then

$\lambda - A$ is injective

$\Leftrightarrow \{ \varphi \in L^2(M, \mu), (\lambda - f(x))\varphi(x) = 0 \text{ a.e.} \Rightarrow \varphi(x) = 0 \text{ a.e.} \}$

$\Leftrightarrow \lambda - f(x) \neq 0 \text{ a.e.} \Leftrightarrow \mu(\{ x \in M \mid f(x) = \lambda \}) = 0.$

$\sigma_p(M_f) = \{ \lambda \in \mathbb{C} \mid \mu(\{ x \in M \mid f(x) = \lambda \}) > 0 \}.$
Let $\lambda \in \mathbb{C} \setminus \sigma_p(M_f)$. So, $M_f - \lambda 1_M$ has an inverse:

$$(M_f - \lambda 1_M)\psi = \varphi \iff (f - \lambda 1_M)\psi = \varphi(x) \text{ a.e.} \iff \psi(x) = \frac{1}{f(x) - \lambda} \varphi \text{ a.e.}$$

So

$$(M_f - \lambda 1_M)^{-1} = M \frac{1}{f - \lambda},$$

with domain

$$D = \{\varphi \in L^2(M, \mu) | M \frac{1}{f - \lambda} \varphi \in L^2(M, \mu)\}.$$  

$M \frac{1}{f - \lambda}$ is bounded $\iff \frac{1}{f - \lambda} \in L^\infty.$

So

$$\varphi(M_f) = \{\lambda \in \mathbb{C} | \exists K > 0 \text{ s.t. } |\lambda - f(x)| \geq K \text{ a.e.}\}.$$
Let $\lambda \in \mathbb{C} \setminus (\sigma_p(A) \cup \varrho(A))$. So \( \mu(\{x \in M \mid f(x) = \lambda\}) = 0 \), but on the other hand \( \mu(\{x \in M \mid |\lambda - f(x)| < \varepsilon\}) > 0 \), for every \( \varepsilon > 0 \). Is the range of \((M_f - \lambda 1_M)\) dense or not? Let for \( n \in \mathbb{N} \),

\[
E_n = \{x \in M \mid f(x) - \lambda \geq \frac{1}{n}\}.
\]

For every \( \psi \in L^2(M, \mu) \), we have \( \chi_{E_n} \psi \) is the image under \((M_f - \lambda 1_M) \); \( \vdots \) of \( \frac{1}{f(x) - \lambda} \chi_{E_n}(x) \psi(x) \in L^2(M, \mu) \). We have \( \chi_{E_n} \psi \) converges pointwise to \( \psi \), so by dominated convergence theorem, we get convergence in \( L^2 \). So, the range of \((M_f - \lambda 1_M)\) is dense. So

\[
\sigma_r(M_f) = \emptyset.
\]
Lemma

Let $A : \mathcal{D}(A) \to X, \mathcal{D} \subset X$, be a closed linear operator in Hilbert space $X$. Then

$$\varrho(A^*) = \varrho(A) \quad \text{and} \quad \sigma(A^*) = \sigma(A).$$
Proposition

Let $A : D(A) \to X, D \subset X$ be self-adjoint. Then

1. $\sigma(A) \subset \mathbb{R}$.
2. $\sigma_r(A) = \emptyset$.
3. If $0$ is not in the spectrum of $A$, then $A^{-1} : A(D(A)) \to X$ is self-adjoint.
Proof:

1. \( \sigma(A) = \overline{\sigma(A)} \).
2. If \( \ker(A - \lambda) = \{0\} \), then

\[
\overline{(A - \lambda)(\mathcal{D}(A))} = \left( (A - \lambda)(\mathcal{D}(A)) \right)^{\perp \perp} = (\ker(A - \lambda))^\perp = X,
\]

thus \( \sigma_r(A) = \emptyset \).
Theorem

Let $A$ be a self-adjoint operator. Then $\lambda \in \sigma(A)$ if and only if there exists a sequence $\{u_n\}_n \subset D(A)$, such that $\|u_n\| = 1$ and $\|(A - \lambda)u_n\| \to 0$ as $n \to +\infty$. 
Proof: Let $\lambda \in \sigma(A)$. Two cases arises:

1. $\ker(A - \lambda) \neq \{0\}$ i.e $\lambda$ is an eigenvalue. Let $f$ be an eigenvector. Then let $u_n = f$ for any $n$ with $\|f\| = 1$.

2. $\ker(A - \lambda) = \{0\}$. Then $\text{Ran}(A - \lambda)$ is dense but not equal to $X$, so $(A - \lambda)^{-1}$ exist but it is unbounded.

Consequently, if there exists a sequence

$\{v_n\}_n \subset \mathcal{D}((A - \lambda)^{-1}), \|v_n\| = 1$ such that

$$\| (A - \lambda)^{-1}v_n \| \rightarrow \infty.$$

Let $u_n = [(A - \lambda)^{-1}v_n], \| (A - \lambda)^{-1}v_n \|^{-1}$, then

$\{u_n\}_n \subset \mathcal{D}(A), \|u_n\| = 1$, and

$$\| (A - \lambda)u_n \| = \|v_n\| \| (A - \lambda)^{-1}v_n \|^{-1} \rightarrow 0.$$
Conversely: Let \( \lambda \in \varrho(A) \). Then there exists \( M > 0 \), such that for any \( u \in X \)

\[
\| R_\lambda(A)u \| \leq M \| u \| .
\]

Let \( v = R_\lambda(A)u \), for \( v \in D(A) \) so that

\[
\| v \| \leq M \| (A - \lambda)v \| ,
\]

and thus no sequence having the properties described can exist.
Definition

$B : \mathcal{D}(B) \to X$ is called $A$ bounded with respect to $A : \mathcal{D} \to X$ densely defined operator if

1. $\mathcal{D}(A) \subset \mathcal{D}(B)$
2. There exists $a, b \in [0, \infty)$; $\forall x \in \mathcal{D}(A)$:

$$\| Bx \| \leq a \| Ax \| + b \| x \| .$$
Kato-Rellich Theorem

**Theorem**

Let $A : \mathcal{D}(A) \to X, \mathcal{D}(A) \subset X$ be a selfadjoint operator on some Hilbert space $X$ and $B : \mathcal{D}(A) \to X$ be symmetric and $A$-bounded with relative bound $< 1$. Then

$$A + B : \mathcal{D}(A) \to X$$

is selfadjoint.
First we note that $A + B : \mathcal{D}(A) \to X$ is symmetric as

$$\forall x, y \in \mathcal{D}(A) : \langle (A + B)x, y \rangle = \langle Ax, y \rangle + \langle Bx, y \rangle = \langle x, Ay \rangle + \langle x, By \rangle = \langle x, (A + B)y \rangle.$$  

Let $x \in \mathcal{D}(A)$ and $\eta \in \mathbb{R}\{0\}$. Then

$$\| (A + i\eta)x \|^2 = \| Ax \|^2 + \eta^2 \| x \|^2.$$  

Implies that for $x = (A + i\eta)^{-1}y, y \in X$:

$$\| A(A + i\eta)^{-1}y \| < \| y \| \text{ and } \| (A + i\eta)^{-1}y \| \leq \frac{1}{|\eta|} \| y \|$$  

$$\Rightarrow \| B(A + i\eta)^{-1}y \| \leq a \| A(A + i\eta)^{-1}y \| + b \| (A + i\eta)^{-1}y \|$$  

$$< a \| y \| + \frac{b}{\eta} \| y \|.$$  

H. Najar  
Introduction to spectral theory of unbounded operators.
So by Neumann Theorem $C = 1 + B(A + i\eta)^{-1}$ is invertible and $\text{range}(C) = X$. As $(A + i\eta)\mathcal{D}(A) = X$, we have

$$X = C(A + i\eta)(\mathcal{D}(A)) = (A + B + i\eta)(\mathcal{D}(A)).$$

Thus $A + B$ is self-adjoint operator.
Remark

It can be proved that if $A$ is essentially selfadjoint operator and $B : \mathcal{D}(A) \rightarrow X$ is symmetric with $A$-bound less than one, then $A + B : \mathcal{D} \rightarrow X$ is e.s.a. and

$$A + B = \overline{A} + \overline{B}.$$ 

Theorem

Let $A$ be a selfadjoint operator, with domain $\mathcal{D}(A)$ and $B$ a compcat operator. Then $A + B$ is a selfadjoint operator on domain $\mathcal{D}(A)$ and

$$\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(A + B).$$
If $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ is real valued. Then

$$H = -\Delta + M_V,$$

is selfadjoint on $\mathcal{D}(-\Delta) = W^{2,2}(\mathbb{R}^3)$ and e.s.a. on $C_0^\infty(\mathbb{R}^3)$.

**Lemma**

$\forall f \in W^{2,2}(\mathbb{R}^3), \forall a > 0, \exists b \in \mathbb{R}$

$$\|f\|_\infty \leq a\|\Delta f\|_2 + b\|f\|_2.$$
Let $a(\cdot, \cdot)$ be a sesquilinear form defined on a dense domain $\mathcal{D}(a)$. We say that $a$ is \textbf{semibounded}, if there exists $m \in \mathbb{R}$ such that

$$a(u, u) \geq m \| x \|^2 \quad \forall u \in \mathcal{D}(a).$$

If the largest $m$ is positive, we say that that is \textbf{definite positive}.

A symmetric operator $S$ is said to be bounded from below if

$$\langle Su, u \rangle \geq m \| u \|^2, \forall u \in \mathcal{D}(S),$$

with some $m \in \mathbb{R}$. 
Remark

The inner product

$$\langle u, v \rangle_a = (1 - m)\langle u, v \rangle + a(u, v),$$

satisfies

$$\| u \|_a \geq \| u \|, \; \forall u \in \mathcal{D}(a).$$
Theorem

Let \((H, \langle \cdot, \cdot \rangle)\) be a Hilbert space and let \(H_1\) be a dense subspace of \(H\). Assume that an inner product \(\langle \cdot, \cdot \rangle_1\) is defined on \(H_1\) in such a way that \((H_1, \langle \cdot, \cdot \rangle_1)\) is a Hilbert space and with some \(m > 0\) we have

\[
m \| f \|^2_1 \leq \| f \|^2, \forall f \in H_1.
\]

Then there exists a unique self-adjoint operator \(T\) on \(H\) such that for \(\mathcal{D}(T) \subset H_1\) and \(\langle Tf, g \rangle = \langle f, g \rangle_1\), for all \(f \in \mathcal{D}(T), g \in H_1\), where \(T\) is bounded from below with lower bound \(m\). The operator \(T\) can be defined by the equalities

\[
\mathcal{D}(T) = \{ f \in H_1 : \exists \tilde{f} \in H, s.t \langle f, g \rangle_1 = \langle \tilde{f}, g \rangle \forall g \in H_1 \}, \quad (11)
\]

and \(Tf = \tilde{f}\), where \(\mathcal{D}(T)\) is dense in \(H_1\) w.r.t. \(\| \cdot \|_1\).
Proof: First we check if such an operator defined by (11) exists since \( H_1 \) is dense \( \bar{f} \) exists and is uniquely determined. The mapping \( f \rightarrow \bar{f} \) is linear and so (11) define a linear operator, we denote it by

\[
T : (H, \langle \cdot, \cdot \rangle) \rightarrow (H, \langle \cdot, \cdot \rangle), \ D(T) \subset H_1.
\]

But we can also define

\[
T_0 : (H_1, \langle \cdot, \cdot \rangle_1) \rightarrow (H, \langle \cdot, \cdot \rangle)
\]

with \( D(T) = D(T_0) \). Then for all \( f \in D(T) \) we have

\[
Tf = T_0 f,
\]

and for all \( f \in H_1 \) we have by (11)

\[
\langle f, g \rangle_1 = \langle \bar{f}, g \rangle = \langle T_0 f, g \rangle.
\]
Also, for all $f \in \mathcal{D}(T), g \in H_1$ we have

$$\langle T_0 f, g \rangle = \langle f, T_0^* g \rangle_1 \iff \langle f, g \rangle_1 = \langle f, T_0^* g \rangle_1 \Rightarrow T_0^* g = g,$$

for all $g \in H_1$ i.e $\mathcal{D}(T_0^*) = H_1$. Furthermore, define

$$J : (H, \langle \cdot, \cdot \rangle) \rightarrow (H_1, \langle \cdot, \cdot \rangle)$$

with $\mathcal{D}(J) = H_1$ and $Jf = f$. Then for all $g \in H_1, f \in \mathcal{D}(T)$ we have

$$\langle f, Jg \rangle_1 = \langle f, g \rangle_1 = \langle f, T_0^* g \rangle_1.$$

Thus $J = T_0^*$.  

H. Najar  
Introduction to spectral theory of unbounded operators.
Assume that $J$ is closed, then $T_0^* = J$ is densely defined. Thus, since $\mathcal{D}(T) = \mathcal{D}(T_0)$, we have that $T$ is densely defined in $H_1$ w.r.t. $\| \cdot \|_1$ and consequently in $H$ w.r.t. $\| \cdot \|$. By (11) we have for all $f, g \in \mathcal{D}(T)$

$$\langle Tf, g \rangle = \langle f, g \rangle_1 = \overline{\langle g, f \rangle_1}$$

$$= \langle Tg, f \rangle \text{ as } g \in \mathcal{D}(T)$$

$$= \langle f, Tg \rangle.$$

So $T$ is symmetric.
Assume that selfadjointness of $T$ follows if $\text{Range}(T) = H$. Let $f \in \mathcal{H}$ be arbitrary. Then

$$g \mapsto \langle f, g \rangle$$

is a continuous linear functional on $H_1$ because

$$|\langle f, g \rangle| \leq \|f\| \|g\| \leq m^{-1/2} \|f\| \cdot \|g\|_1.$$  

Therefore there exists an $\overline{f} \in H_1$ such that

$$\langle f, g \rangle = \langle \overline{f}, g \rangle_1,$$

for all $g \in H_1$ by Riesz Theorem. This means that $\overline{f} \in \mathcal{D}(T)$ and $f = T\overline{f}$. The semi-boundedness follows from

$$\langle Tf, f \rangle = \langle f, f \rangle_1 \geq m \|f\|^2, \forall f \in \mathcal{D}(T).$$
Uniqueness. If $S$ satisfies $\mathcal{D}(S) \subset H_1$ and

$$\langle Sf, g \rangle = \langle f, g \rangle_1.$$ 

Then $S = T|_{\mathcal{D}(S)}$, i.e $S \subseteq T \subseteq T^* \subseteq S^*$. If $S$ is self adjoint this implies

$$S = T.$$
Theorem

Assume $H$ is a Hilbert space. $\mathcal{D}$ is a dense subspace of $H$ and $s(\cdot, \cdot)$ is a semi-bounded symmetric sesquilinear form on $\mathcal{D}$ with lower bound $m$. Let $\| \cdot \|_s$ be compatible with $\| \cdot \|$. Then there exists a unique semi-bounded selfadjoint operator $T$ with lower bound $m$ such that $\mathcal{D}(T) \subseteq H$ and $\langle T f, g \rangle = s(f, g)$ for all $f \in \mathcal{D} \cap \mathcal{D}(T), g \in \mathcal{D}$. We have

$$\mathcal{D}(T) = \{ f \in H_s : \exists \bar{f} \in H, \text{s.t.} s(f, g) = \langle \bar{f}, g \rangle \forall g \in \mathcal{D} \}. \quad (12)$$

Where $T f = \bar{f}$ for $f \in \mathcal{D}(T). H_s$ is the completion of $(\mathcal{D}, \| \cdot \|_s)$. 

H. Najar
Introduction to spectral theory of unbounded operators.
Replace \((H_1, \langle \cdot, \cdot \rangle)\) by \((H_s, \langle \cdot, \cdot \rangle)\) in the last theorem. Then we obtain exactly one self adjoint operator \(T_0\) such that \(\mathcal{D}(T_0) \subseteq H_s\) and

\[
\langle T_0 f, g \rangle = \langle f, g \rangle_s = (1 - m) \langle f, g \rangle + s(f, g),
\]

for all \(f \in \mathcal{D}(T_0), g \in H_s\). Also, \(T_0\) is semi-bounded with lower bound 1 because

\[
\langle f, T_0 f \rangle = \langle f, f \rangle_s = (1 - m) \langle f, f \rangle + s(f, f) \geq (1 - m) \| f \|^2 + m \| f \|^2 = \| f \|^2.
\]
Define $T = T_0 - (1 - m)$. Then

$$\langle Tf, f \rangle = \langle (T_0 - (1 - m))f, f \rangle$$

$$= \langle T_0 f, f \rangle - \langle (1 - m)f, f \rangle$$

$$\geq \| f \|^2 - \| f \|^2 + m \| f \|_s^2$$

$$= m \| f \|_s^2 .$$
1 $\mathcal{D}(T) \subseteq H_s$ follows easily from $\mathcal{D}(T_0) \subseteq H_s$. This because shifting an operator by a constant does not change the domain.

2

$$\langle Tf, g \rangle = \langle [T_0 - (1 - m)]f, g \rangle$$
$$= \langle T_0f, g \rangle - \langle (1 - m)f, g \rangle$$
$$= (1 - m)\langle f, g \rangle + s(f, g) - \langle (1 - m)f, g \rangle$$
$$= s(f, g)$$

for all $f \in \mathcal{D} \cap \mathcal{D}(T_0), g \in \mathcal{D}$. **Uniqueness** of $T$ follows from uniqueness of $T_0$. 

H. Najar
Introduction to spectral theory of unbounded operators.
Theorem

Let $S$ be a semi-bounded symmetric operator with lower bound $m > 0$. Then there exists a semi-bounded self-adjoint extension of $S$ with lower bound $m$. If we define

$$s(f, g) = \langle Sf, g \rangle, \forall f, g \in \mathcal{D}(S),$$

for $H_s$, the completion of $(\mathcal{D}(S), \| \cdot \|_s)$ then we have the operator $T$ defined by

$$\mathcal{D}(T) = \mathcal{D}(S^*) \cap H_s$$

and $Tf = S^*f$ for all $f \in \mathcal{D}(T)$ is a selfadjoint extension of $S$ with lower bound $m$. The operator $T$ is the only selfadjoint extension of $S$ having the property $\mathcal{D}(T) \subseteq H_s$. 

H. Najar

Introduction to spectral theory of unbounded operators.
**Proof:** By the last theorem we know there exists a unique selfadjoint operator $T$ with $\mathcal{D}(T) \subseteq H_s$ and

$$\langle Tf, g \rangle = s(f, g) = \langle Sf, g \rangle, \forall f \in \mathcal{D}(S) \cap \mathcal{D}(T),$$

and $m$ is lower bound for $T$. We have by (12)

$$\mathcal{D}(T) = \{ f \in H_s : \exists \bar{f} \in H, \bar{s}(f, g) = \langle \bar{f}, g \rangle \forall \langle, \forall g \in \mathcal{D}(S) \}. $$
Let \((f_n)_n \in \mathcal{D}(S)\) such that
\[
\|f_n - f\| \to 0.
\]

Then we obtain
\[
\bar{s}(f, g) = \lim_{n \to \infty} \bar{s}(f_n, g) = \lim_{n \to \infty} (\langle f_n, g \rangle_s - (1 - m)\langle f_n, g \rangle) \\
= \lim_{n \to \infty} ((1 - m)\langle f_n, g \rangle + s(f_n, g) - (1 - m)\langle f_n, g \rangle) \\
= \lim_{n \to \infty} s(f_n, g) = \lim_{n \to \infty} \langle Sf_n, g \rangle \\
= \lim_{n \to \infty} \langle f_n, Sg \rangle = \lim_{n \to \infty} \langle f, Sg \rangle,
\]

because \(\| \cdot \|_s\) is compatible with \(\| \cdot \|\). So we can replace \(\bar{s}(f, g)\) with \(\langle f, Sg \rangle\).
We have to show $T$ is an extension of $S$:

1. From definition of $\mathcal{D}(T) = \mathcal{D}(S^*) \cap H_s$. Also $T = S^* \big|_{\mathcal{D}(T)}$.

2. Since $S$ is symmetric then $S \subseteq S^*$. Also $S \subseteq H_s$ by construction. Thus

$$\mathcal{D}(S) \subseteq \mathcal{D}(S^*) \cap H_s = \mathcal{D}(T).$$

Furthermore, since $S$ is symmetric then $S = S^* \big|_{\mathcal{D}(S)}$ which means that, by 1, $S = T \big|_{\mathcal{D}(S)}$. Thus we have $S \subseteq T$. 

H. Najar

Introduction to spectral theory of unbounded operators.
(Uniqueness) Let $A$ be an arbitrary self adjoint extension of $S$ such that $\mathcal{D}(A) \subseteq H_s$. Then since $S \subseteq A$ we have $A \subseteq S^*$ which means $\mathcal{D}(A) \subseteq \mathcal{D}(S^*)$ and $A = S^* |_{\mathcal{D}(A)}$. Also,

$$\mathcal{D}(T) = \mathcal{D}(S^*) \cap H_s,$$

which means $\mathcal{D}(A) \subset \mathcal{D}(T)$ and so

$$A = S^* |_{\mathcal{D}(A)} = T |_{\mathcal{D}(A)}.$$

Thus $A \subseteq T$ which implies $T = T^* \subseteq A^* = A$ and so $A = T$. 

H. Najar
Introduction to spectral theory of unbounded operators.
Lemma

Let $T$ be a self-adjoint operator and densely defined. For $\lambda \in \mathbb{C} \setminus \mathbb{R}$, the operator $R_\lambda$ is everywhere defined on $X$, and the norm is estimated by

$$\|R_\lambda\| \leq \frac{1}{|\text{Im}\lambda|}.$$
Proof: For $\lambda = x + iy$ and $v \in \mathcal{D}(T),$

$$| (T - \lambda)v |^2$$

$$= | (T + x)v |^2 + \langle (T - x)v, iyv \rangle + \langle iyv, (T - x)v \rangle + y^2 | v |^2$$

$$= | (T + x)v |^2 - iy \langle (T - x)v, v \rangle + iy \langle v, (T - x)v \rangle + y^2 | v |^2$$

$$= | (T - x)v |^2 + y^2 | v |^2 \geq y^2 | v |^2 .$$

Thus, for $y \neq 0, (T - \lambda)v \neq 0.$ On $(T - \lambda)\mathcal{D}(T),$ there is an inverse $R_{\lambda}$ of $T - \lambda,$ and for $w = (T - \lambda)v, v \in \mathcal{D}(T)$

$$| w | = | (T-\lambda)v | \geq | y | \cdot | v | = | y | \cdot | R_{\lambda}(T-\lambda)v | = | y | \cdot | R_{\lambda}w |$$

which gives

$$\| R_{\lambda}w \| \leq \frac{1}{| Im\lambda |} \cdot \| w \| \text{ (for } (T - \lambda)v, v \in \mathcal{D}(T)) .$$
Thus, the operator norm on \((T - \lambda)\mathcal{D}(T)\) satisfies \(\| R_\lambda \| \leq \frac{1}{\text{Im} \lambda}\) as claimed. It remains to show that \((T - \lambda)\mathcal{D} = X\), the hole space. If
\[
\langle (T - \lambda)v, w \rangle = 0, \quad \forall v \in \mathcal{D}(T).
\]
So \(T - \lambda\) can be defined on \(w\) as \((T - \lambda)^*w = 0\), this gives \(Tw = \overline{\lambda}w\), so \(w = 0\). Thus, \((T - \lambda)\mathcal{D}(T)\) is dense in \(X\). As \(T\) is closed we get it is equal to \(X\).
Definition

Let $x \in \mathcal{A}$ and $\lambda$ an isolated point of $\sigma(x)$. Let $\Gamma_{\lambda_0}$ be an admissible contour i.e a closed contour around $\lambda_0$ such that the closure of the region bounded by $\Gamma_{\lambda_0}$ intersects $\sigma(x)$ only at $\lambda_0$,

$$P_{\lambda_0} = \frac{1}{2\pi i} \oint_{\Gamma_{\lambda_0}} R_\lambda(A) d\lambda,$$

is called Riesz integral for $x$ and $\lambda_0$. 

H. Najar
Introduction to spectral theory of unbounded operators. 

Perturbation theory
Lower bounded operators and quadratic forms

Spectrum of unbounded operators on Hilbert spaces
Proposition: Let $P_{\lambda_0}$ be a Riesz integral for $x$ and $\lambda_0$.

1. $P_{\lambda_0}$ is a projection.
2. $\text{Ker}(x - \lambda_0) \subset \text{Ran} P_{\lambda_0}$.
3. If $\mathcal{A}$ is a Hilbert space and $x$ is self adjoint, then $P_{\lambda_0}$ is orthogonal projection onto $\text{ker}(x - \lambda_0)$. 

H. Najar
Introduction to spectral theory of unbounded operators.
**Proof:** (1) Let $\Gamma_{\lambda_0}$ and $\tilde{\Gamma}_{\lambda_0}$ be two admissible contours for defining $P_{\lambda_0}$, we suppose that $\Gamma_{\lambda_0}$ is contained in the interior of the region bounded by $\tilde{\Gamma}_{\lambda_0}$.

\[
P_{\lambda_0}^2 = \frac{1}{(2\pi i)^2} \oint_{\Gamma_{\lambda_0}} d\lambda \oint_{\tilde{\Gamma}_{\lambda_0}} d\mu R_{\lambda}(x)R_{\mu}(x)d\mu
\]

\[
= \frac{1}{(2\pi i)^2} \oint_{\Gamma_{\lambda_0}} d\lambda \oint_{\tilde{\Gamma}_{\lambda_0}} (\mu - \lambda)^{-1}[R_{\lambda}(x) - R_{\mu}(x)]d\mu.
\]

Using the residue theorem, we get:

\[
\oint_{\Gamma_{\lambda_0}} d\lambda \oint_{\tilde{\Gamma}_{\lambda_0}} (\mu - \lambda)^{-1}R_{\lambda}(x)d\mu = 2\pi i \oint_{\Gamma_{\lambda_0}} R\lambda(x)d\lambda.
\]

For the second integral we get that

\[
\oint_{\Gamma_{\lambda_0}} d\lambda \oint_{\tilde{\Gamma}_{\lambda_0}} (\mu - \lambda)^{-1}R_{\mu}d\mu = \oint_{\tilde{\Gamma}_{\lambda_0}} R_{\mu}(x)d\mu \oint_{\Gamma_{\lambda_0}} (\mu - \lambda)^{-1}d\lambda = 0.
\]
(2). Let \( f \in \ker(x - \lambda_0) \). Then for \( \lambda \neq \lambda_0 \)

\[
(x - \lambda_0)^{-1}f = (\lambda_0 - \lambda)^{-1}f.
\]

We show that \( P_{\lambda_0}f = f \), so \( f \in \text{Ran}P_{\lambda_0} \). By the definition of \( P_{\lambda_0} \) we find that

\[
P_{\lambda_0}f = \frac{1}{2\pi} \oint_{\Gamma_{\lambda_0}} (x - \lambda)^{-1}f d\lambda \quad (15)
\]

\[
= \frac{1}{2\pi} \oint_{\Gamma_{\lambda_0}} \oint_{\Gamma_{\lambda_0}} (\lambda_0 - \lambda)^{-1}f d\lambda = f \quad (16)
\]

(3) Let \( x \) be an Hilbert space and suppose that \( x = x^* \) (Exercise: show that \( P_{\lambda_0} = P_{\lambda_0}^* \)). We must show now that \( \text{Ran}P_{\lambda_0} \subset \ker(x - \lambda_0) \). We compute

\[
(x - \lambda_0)P_{\lambda_0} = \frac{1}{2\pi} \oint_{\Gamma_{\lambda_0}} (x - \lambda_0)(x - \lambda)^{-1}d\lambda \quad (17)
\]

H. Najar

Introduction to spectral theory of unbounded operators.
Consider $U_{\lambda_0}$ denote the interior of $\Gamma_{\lambda_0}$. On $U_{\lambda_0}\setminus\{\lambda_0\}$, the operator $(\lambda - \lambda_0)(x - \lambda)^{-1}$ is analytic, operator and satisfies

$$|\lambda_0 - \lambda\|(x - \lambda)^{-1}\| \leq |\lambda_0 - \lambda|d(\lambda, \sigma(x))^{-1}. \quad (19)$$

We can choose $\Gamma_{\lambda_0}$, so that $\lambda_0$ is the closest point of $\sigma(x)$ to $\Gamma_{\lambda_0}$. So $|\lambda_0 - \lambda\|(x - \lambda)^{-1}\| \leq 1$ and this function is uniformly bounded on $U_{\lambda_0}\setminus\{\lambda_0\}$. It follows that $(\lambda_0 - \lambda)(x - \lambda)^{-1}$ extends to analytic function on $U_{\lambda_0}$ so by Cauchy theorem the integral(19) vanishes. This gives that $\text{Ran}P_{\lambda_0} \subset \text{Ker}(x - \lambda_0)$. 
References:
Thanks