Given a finite Borel measure on $\mathbb{R}$, its Fourier transform is defined as
\[ \hat{\mu}(\xi) = \int_{\mathbb{R}} e^{2\pi i x \xi} d\mu(x). \]

The decay properties of $\hat{\mu}(\xi)$ as $|\xi| \to +\infty$ give crucial information about $\mu$.

We say that $\hat{\mu}(\xi)$ has polynomial decay if there exist $C_\sigma, \sigma > 0$ such that
\[ |\hat{\mu}(\xi)| \leq C_\sigma |\xi|^{-\sigma}. \]

The supreme of all these $\sigma$ is called Fourier dimension of the measure $\mu$ : $\dim_F(\mu)$.

Then $\dim_F(\mu) > 0$ if and only if $\hat{\mu}$ has polynomial decay.

### Self-similar measures

Given a finite set of contracting similarity maps of $\mathbb{R}^d$, $S_1, \ldots, S_m$, and weights $p_1, \ldots, p_m$ such that $p_1 + \cdots + p_m = 1$, there exists a unique probability Borel measure $\mu$ on $\mathbb{R}^d$ such that
\[ \mu(A) = \sum_{i=1}^m p_i S_i^{-1} A, \quad \text{for all Borel sets } A \subseteq \mathbb{R}^d. \]

$(S_1, \ldots, S_m; (p_1, \ldots, p_m))$ is an iterated function system of similarities with weights or IFS$w$ and $\mu$ is the invariant measure or attractor of the IFS$w$.

The simplest class of self-similar measures is Bernoulli convolutions: $d = 1$, $m = 2$, $\lambda \in (0, 1)$ and
\[ S_1(x) = \lambda x - 1, \quad S_2(x) = \lambda x + 1, \quad p_1 = p_2 = \frac{1}{2}. \]

The attractor of this IFS$w$ is called Bernoulli convolution $\mu_\lambda$. The special case $\lambda = 1/3$ yields the Cantor-Lebesgue measure.

### Goal

Give explicit bounds for the polynomial decay of the Fourier transform of self-similar measures outside a small set of exceptions.

### Previous work

- Erdős [3], for Bernoulli convolutions, proved that $\dim_F(\mu_\lambda) > 0$ for almost all $\lambda$ but there is an infinite measurable set of $\lambda$s such that $\hat{\mu}_\lambda(\xi)$ does not even tend to zero when $|\xi| \to +\infty$.
- Kalpakis [4] used the Erdős argument to show that $\dim_F(\mu_\lambda) > 0$ for all $\lambda$ outside of zero Hausdorff-dimensional set of exceptions.
- In [5], for $d=1$, the authors showed that certain Bernoulli convolutions associated to algebraic numbers have at least logarithmic decay.
- Kaufman [6] proved that if $F$ is any C1$^\alpha$ diffeomorphism of $\mathbb{R}$ such that $\lambda^n > 0$ then $\dim_F(\mu_\lambda) > 0$ where $F_\lambda(A) = \mu(F^{-1} A)$ for all Borel sets $A \subseteq \mathbb{R}$. He provided his result for Bernoulli convolutions with $\lambda \in (0, 1/2)$.
- The study of the Fourier decay of self-similar measures has become relevant since it is a key component of the study of continued fractions of self-similar measures.

### Results

Let $\mu_\lambda^{p,c}$ be the self-similar measure for the IFS$w$ $(a_1, \ldots, a_n)$ with weights $p = (p_1, \ldots, p_m)$, where $\lambda = (a_1, \ldots, a_n) \in (0, 1)$. In this case
\[ \hat{\mu}_\lambda^{p,c}(\xi) = \lambda^\frac{C_1}{\log(1/\xi)}, \]

where $C_1 = \lambda^\frac{\log(2) - \log(\log(2))}{\log(1/\lambda)}$.

Theorem. Let $\mu$ be a Bernoulli convolution on $\mathbb{R}$ such that $\lambda^n > 0$ and let $\mu = \mu_\lambda^{p,c}$ be a homogeneous self-similar measure on $\mathbb{R}$ which is not a single atom. Then there exist $\sigma = \sigma(\mu) > 0$ (independent of $F$) and $C = C(\mu) > 0$ such that
\[ |\hat{\mu}(\xi)| \leq C |\xi|^{-\sigma}. \]

We underline that the value of $\sigma$ is effective.

As an example, we obtain that if $\mu$ is the Cantor-Lebesgue measure on the middle-third Cantor set, then even though $\hat{\mu}(\xi)$ does not decay as $\mu \to +\infty$, for $F \mu$ we have a uniform explicit decay:

Corollary. Let $\mu$ be the Cantor-Lebesgue measure. Then for every $C^2$ function $F : \mathbb{R} \to \mathbb{R}$ such that $\lambda^n > 0$ there exists a constant $C_{\mu} > 0$ such that
\[ |\hat{F}(\xi)| \leq C_{\mu} |\xi|^{-\sigma}. \]

Using the above results we can obtain estimates for the dimension of Bernoulli convolutions.

### l^p dimension of convolutions

Definition. Let $p \in (1, +\infty)$. The $l^p$-dimension of the measure $\mu$ is defined as
\[ \dim_{l^p}(\mu) = \lim_{r \to 0} \frac{\log \int \mu(\xi) \xi \cdot r \cdot x \in [-r, r])}{\log r}. \]

For $p = \infty$, we define
\[ \dim_{l^p}(\mu) = \lim_{r \to 0} \frac{\log \left( \int \mu(\xi) \xi \cdot r \cdot x \in [-r, r]) \right)}{\log r}. \]

The function $p \to \dim_{l^p}(\mu)$ is non-increasing for all probability measures.

Remark. Let $\mu = \mu_\lambda^{p,c}$ be as above. Given any $\alpha > 0$, there is $\sigma(\mu, \alpha) > 0$ such that the following holds: let $\nu$ be any Borel probability measure with $\dim_{l^p}(\nu) \leq 1 - \alpha$. Then
\[ \dim_{l^p}(\mu(\nu)) \leq \dim_{l^p}(\nu) + \alpha. \]

More precisely, one can take $\sigma = 2\delta$, where $\delta = \delta(c, p, \alpha)$ is such that the value of $\delta = \delta(c, p, \alpha)$ given in Proposition (a) satisfies $\alpha < 2\delta = \delta$.

### Dimension of Bernoulli convolutions

Theorem. Let $\mu_\lambda^{p,c}$ be the biased Bernoulli convolution of parameter $\lambda \in (0, 1)$ and weight $p \in (0, 1)$. Then, for every $p \in (0, 1/2)$ there is $C(\mu) > 0$ such that
\[ \inf \left\{ \mu(\nu) : \dim_{l^p}(\mu(\nu)) \geq 1 - (1 - \lambda) \log(1/(1 - \lambda)) \right\} > C(\mu). \]

We present two corollaries. The first is a corollary of the proof rather than the statement. For the case of unbiased Bernoulli convolutions we are able to obtain an improved lower bound.

Corollary. There is an absolute constant $C > 0$ such that
\[ \dim_{l^p}(\mu_\lambda^{p,c}) \geq 1 - (1 - \lambda)^2 \log(1/(1 - \lambda)). \]

For $\dim_{l^p}(\mu_\lambda^{p,c})$ we obtain the same lower bound as in the above theorem and we can conclude that

Corollary.
\[ \lim_{\lambda \to 0} \dim_{l^p}(\mu_\lambda^{p,c}) = 1 \]

with a quantitative rate.

### Experiments

- Consider the attractor $\mu_\lambda^{p,c}$ of the IPS $(\mathbb{R} \to F, \mathbb{R}_p + F)$ with weights $(1/2, 2/2)$, where $\lambda \in (0, 1)$, $O$ is an orthogonal map on $\mathbb{R}^d$ and $F$ is the identity (that is, $a$ is a generalization of Bernoulli convolutions for dimension $d > 1$) and study the Fourier decay of $\mu_\lambda^{p,c}$.
- Given $\lambda_1, \lambda_2 \in (0, 1)$, consider the attractor $\mu_\lambda^{p,c}$ of the IPS $(\mathbb{R} \to F, \mathbb{R}_p + F)$ with weights $(1/2, 1/2)$ (that is, a non-homogeneous version of Bernoulli convolution) and study the Fourier decay of $\mu_\lambda^{p,c}$.

### References